CONDITIONS FOR GLOBAL SYNCHRONIZATION IN LATTICES OF CHAOTIC ELEMENTS WITH LOCAL CONNECTIONS

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We investigate phenomena on the edge of spatially homogeneous chaotic mode and spatiotemporal chaos in a lattice of chaotic 1-D maps with local connections. We show that spatially homogeneous chaotic mode cannot exist in a lattice with local connections if the Lyapunov exponent λ of the isolated chaotic map is greater than some critical positive value. We propose a few schemes that make spatial synchronization possible in large lattices. If the idea of only local connections is abandoned, the number of connections necessary for synchronization dramatically decreases to three per node. We also propose a model of a lattice with an external pacemaker, where we find a spatially homogeneous mode synchronous with the pacemaker, as well as different from the pacemaker mode.

1. Introduction

The idea of this report comes from complexity that investigates behavior of the systems composed, as a rule, of many interacting elements [Gell-Mann, 1995]. We are especially interested in the behavior at the boundary between the regular motion and chaos. As was noticed, processes similar to evolution or information processing can take place at this boundary often called the "edge of chaos" [Langton, 1986], [Gutowitz, 1995/1996]. The most remarkable example of a system on the edge of chaos is the Earth with the main processes taking place in a thin layer between the globe and the space.

An attractive model of the edge of chaos is synchronization. Indeed, synchronization is abrupt simplification of the system dynamics with respect to the "true" multidimensional chaos of the system elements in the absence of interaction, it essentially simplifies the system dynamics by means of decreasing the attractor dimension. It is directly determined by the degree of the system couplings: the synchronization is turned on only in the presence of sufficiently strong coupling. As follows from investigations of the fine structure of chaotic synchronization attractors, perturbations can lead to on-off intermittency, or to local or global mode changes, etc [Heagy *et al.*, 1994], [Maystrenko & Kapitaniak, 1996].

On-off intermittency, accompanied by desynchronization bursts, in turn, is a form of avalanches characteristic of the systems on the edge of chaos.

In this report, we investigate the problem of global synchronization of lattices of chaotic maps and find the edge of spatiotemporal and spatially homogeneous chaos in such systems. Although the study of coupled map lattices has a long and rich history, the researchers mostly concentrated their efforts on the behavior of the patterns observed in the lattices (mostly 1-D lattices) or on the lattice behavior with variations of the map parameters, etc. Here we investigate only one of the variety of the lattice modes (spatially synchronous mode), and use rigorous analytic and numerical methods.

2. Method for Calculation of the Stability of Synchronous Mode [Dmitriev *et al.*, 1995, 1996, 1997]

The problem of the stability of the global synchronous mode in ensembles of maps coupled with the mean field method (dissipatively) was successively solved a few years ago by Dmitriev *et al.* [1995]. A detailed description of the method along with discussion of the related problem of "on–off" intermittency is given in [Dmitriev *et al.*, 1996, 1997].

They considered an ensemble of P coupled equal 1-D maps

$$x_i(t+1) = f\left(x_i(t) + \sum_{j\neq i} \alpha_{ij}\left(x_j(t) - x_i(t)\right)\right) = f\left(\sum_{j=1}^{P} a_{ij} x_j\right),$$
$$i = 1, \ldots, P, \qquad (1)$$

where a_{ij} is the strength with which jth map acts on ith one, f is the mapping function, and $\mathbf{A} = \; a_{ij} \;$ is a $P \times P$ matrix of the couplings of ensemble (1). The synchronous mode of ensemble (1), i.e., a spatiotemporal mode, at which the mode of every map coincides with that of the isolated map, always exists regardless of the mapping function f, if the sum of each row of **A** is equal to one, i.e., if for each node of ensemble (1) the weighed field of all other nodes plus the weight of the node itself make unity. This statement can easily be verified directly.

According to this method, to estimate the stability of the synchronous mode, one has to estimate the stability of the motion transversal to attractor of the synchronous mode, located on the main diagonal of P-dimensional hipercube. As was shown in [Dmitriev *et al.*, 1995], the necessary stability conditions could be calculated directly if the eigenvalues of matrix **A** were known. The exact (in linear approximation) condition for the stability of the global synchronous mode with respect to small perturbations is

$$\lambda_T = \lambda + \ln(|e_M|) < 0, \qquad (2)$$

where λ_T is the maximum Lyapunov exponent of the motion in the invariant plane transversal to the system attractor, λ is the isolated map Lyapunov exponent, and e_M is one of the eigenvalues of **A**. To obtain e_M, one has to exclude one eigenvalue $e = 1$ from the set of **A** matrix eigenvalues, and then to choose the one with the maximum magnitude.

Universal condition (2) is applicable to any discrete-time systems making up ensemble (1). In the case of multidimensional systems, λ means the maximum Lyapunov exponent of the isolated system.

3. Model of a Lattice with Local Connections

The concept of an ensemble of chaotic elements that come to synchronization through local interactions is investigated in a model of a 2-D $M \times N$ lattice composed of locally coupled identical 1-D maps. The maps are the logistic maps $x(t+1) = f(x(t)) = \mu x(1-x)$ with μ set at $\mu = 4$. The nodes x_{ij} of the lattice are connected only to the nearest neighbors, so each map is coupled to, e.g., 8 nearest maps. This connection is described by the following relation

$$x_{ij}(t+1) = f\left((1-\alpha) x_{ij} + \alpha \sum_{m=1,n=1}^{3} T_{mn} x_{i+m-2,\, j+n-2} \right). \qquad (3)$$

The dynamics of each node is determined by its own dynamics and by a weighed effect of its closest neighbors. The boundary conditions are periodic: $x_{(M+1)j} = x_{1j}$ and $x_{i(N+1)} = x_{i1}$. The neighborhood is determined by a weight template **T** of the kind of $\mathbf{T}_{\mathrm{Full}} = \begin{pmatrix} 1/8 & 1/8 & 1/8 \\ 1/8 & 0 & 1/8 \\ 1/8 & 1/8 & 1/8 \end{pmatrix}$, similar to those used in cellular neural networks (CNNs) [Chua & Yang, 1988], where the central entry (0) corresponds to node x_{ij}, while all other (nonzero) entries are the weights of those neighbor nodes that act on the current node x_{ij}. The sum of the template entries is always made equal to unity. This template corresponds to 3×3 neighborhood (or the neighborhood of radius 1), however we also tried larger neighborhoods and complex templates with unequal weights.

We investigated the system behavior as a function of the form and strength of the local connections and of the lattice dimensions by means of iterating it, starting from random initial conditions of the lattice nodes. To visualize the node state x_{ij}, a value in the range [0, 1], we depicted it by a cell with a color from the gray-scale palette with black corresponding to zero and white to one.

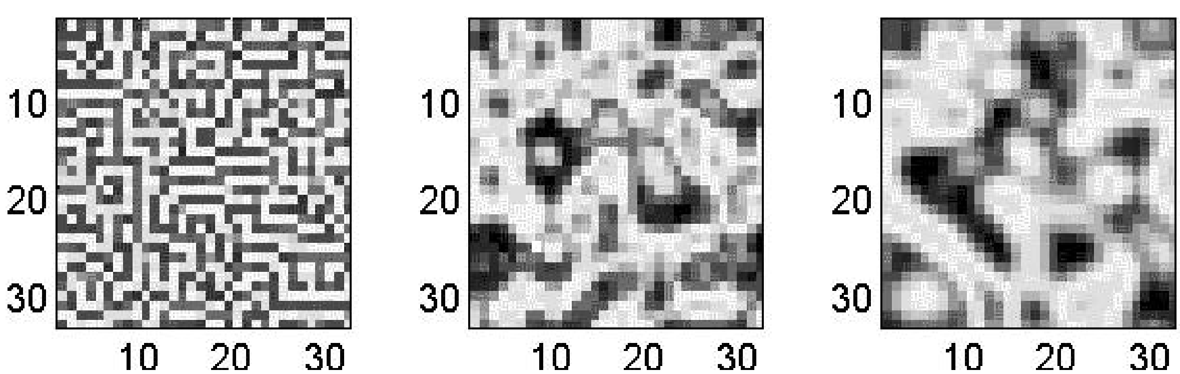

Fig. 1 Clusters in the lattices with the coupling $\alpha = 0.2$, 0.5, and 0.9, respectively. The brightness of a cell corresponds to the value of the corresponding node variable.

As was found, the global synchronous mode is unstable in sufficiently large lattices. Instead, a cluster structure, chaotic in time and space, is observed in the case of nonzero coupling. This cluster structure was many times described earlier, and a good review of the cluster behavior can be found at Kaneko [1989], where a vast list of related references is also given. The observed clusters comprise locally synchronous lattice regions. The nodes within a cluster behave almost synchronously for some time, though their motion is chaotic in time domain. The cluster boundaries are permanently blurred; some clusters disappear, and others are born. Though we are not interested here in the cluster behavior, note that the cluster dimensions seem to depend on the value of the coupling α between the nodes, as can be seen from Fig. 1, where snapshots for 32×32 lattices with different coupling strengths are presented. As α increases from 0 to 1, average cluster dimensions slightly increase (from one to approximately five), but global spatial synchronization is not observed at any coupling strength.

However, in matrices with the dimensions of 5×5 or smaller, global synchronization *is* observed. In order to explain this effect, consider the necessary conditions for the stability of the spatially homogeneous mode using the theory described above.

In order to construct the coupling matrix **A** for an $M \times N$ lattice of locally coupled maps, we enumerate the lattice nodes from 1 to $P = M \times N$ in a regular way, e.g., row by row from top to bottom. One can show that the order of the enumeration can be arbitrary, because it doesn't change the set of **A** matrix eigenvalues. The coupling matrix **A** for P nodes has the size of $P \times P$, and due to only local connections it is sparse, i.e., most of its elements are zeros. A row of matrix **A** corresponds to a node of the lattice. As follows from the relation for local connections (3), the main diagonal of matrix **A** is built of the elements a_{ii} describing the self-action of the node, i.e., $(1-\alpha)$. Each row i of **A** also contains m entries $a_{ij} = \alpha/m$ related to ith node connections to its nearest neighbors. Other elements of the row are null.

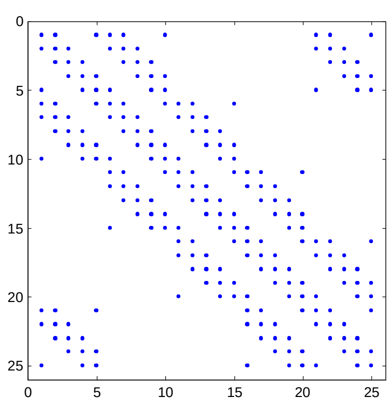

Fig. 2 Coupling matrix for 5×5 lattice. Only nonzero elements are shown.

In Fig. 2, the coupling matrix is schematically shown for 5×5 lattice with the weight template $\mathbf{T}_{\mathrm{Full}}$. Obviously, the coupling matrix becomes more and more sparse with increasing lattice size. Note, that in the case of ensembles of globally coupled maps, the related coupling matrix is full [Dmitriev *et al.*, 1995, 1997].

The stability of the synchronous mode is calculated according to relation (2). In our case, isolated maps of the lattice are chaotic, so $\lambda > 0$. Numerical research shows that global synchronization is impossible in the case of weak couplings between the nodes. As the coupling increases, the synchronous mode becomes possible (i.e., stable) in 3×3 lattice at $\alpha \approx 0.44$, in 4×4 lattice at $\alpha \approx 0.67$, and only at $\alpha \approx 0.96$ in 5×5 lattice (Fig. 3). In larger lattices global spatial synchronization is impossible at couplings $\alpha \in [0, 1]$. These results can explain the increase of the typical cluster size with increasing α in large lattices (will be reported elsewhere).

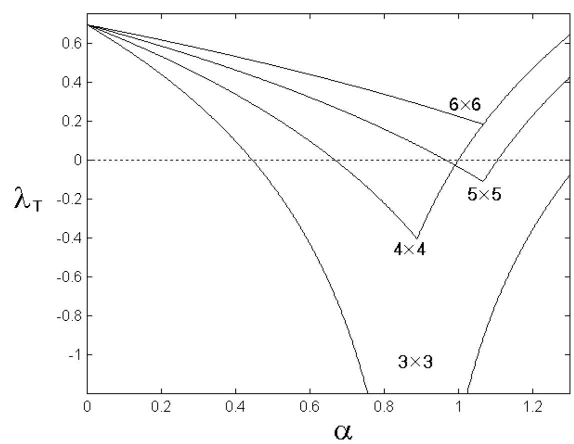

Fig. 3. Transversal Lyapunov exponent as a function of the local coupling α for different lattices

4. Global Synchronous Mode in Large Lattices

As was shown above, in large lattices of chaotic maps with only local connections, the global synchronous mode is impossible. However, an analysis of stability condition (2) gives some mechanisms that can provide this mode in large lattices. Here we discuss four of them. The first is a change of the isolated map dynamics. The other three appear if we reject the idea of only local connections: an increase of the connection "depth", nonlocal connections, and introduction of a special control node (a pacemaker).

4.1. Tuning the chaotic mode

This way to the stable synchronous mode in large lattices consists of a lowering the "chaoticity " of the lattice maps. Indeed, as follows from stability condition (2), the Lyapunov exponent λ_T of the motion in the invariant plane transversal to the synchronization attractor is made up of two terms: the Lyapunov exponent λ of the isolated map (usually a constant) and a term $\ln(|e_M|)$ associated with the topology of the system connections (we shall call it the "topological" term). Closer investigation shows that the second term is negative (or at least zero) practically for all topologies of the lattice node connections. To illustrate this statement, subtract a constant level $\lambda \approx 0.7$ from the dependencies of λ_T in Fig. 3 and obtain the topological term as a function of the coupling strength in different lattices. Both for globally and locally connected lattices, the topological term is negative at the coupling strengths within a "reasonable" range of $\alpha \in [0, 1]$. At stronger $\alpha \sim 1.5 - 2$, the assumptions leading to condition (2) can fail, so we don't operate in this range.

If the mode of the isolated map is regular, i.e., $\lambda < 0$, then the synchronous mode of any lattice is stable. If the lattice nodes are chaotic, i.e., $\lambda > 0$, the synchronous mode will be stable only if the collective interaction of the lattice nodes will be able to compensate the instability of the isolated node dynamics.

If it doesn't happen, as takes place in large lattices, then in order to observe the global synchronization, we have to decrease λ, i.e., the degree of chaoticity of the lattice nodes, which means to change the dynamic mode.

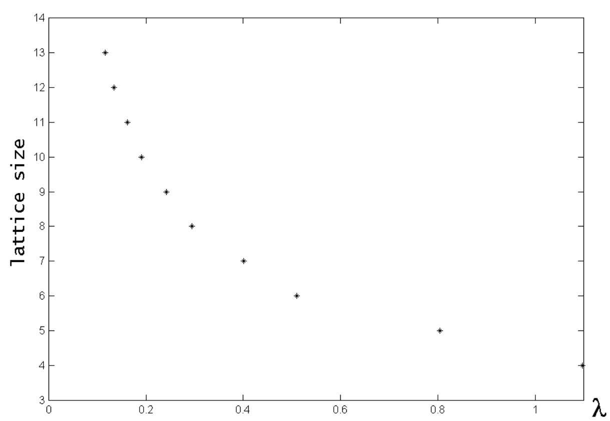

Fig. 4. Maximum admissible Lyapunov exponent of the isolated map as a function of the lattice size.

In Fig. 4, we depicted the relationship between the maximum Lyapunov exponent of the isolated map, at which the stable global synchronous mode is still observable in the lattice, and the linear size of the lattice with the neighborhood template $\mathbf{T}_{\mathrm{Full}}$. If we denote a linear dimension of the lattice by N, then, using the data from the plot, we can approximate the relationship asymptotically for large N as $\lambda \propto N^{-1.9}$, or $N \propto \lambda^{-0.5}$. We can find from this plot that, e.g., if the isolated map dynamics has Lyapunov exponent $\lambda = \ln 2 \approx 0.69$, as is the case of logistic map with $\mu = 4$, the synchronous mode is possible in 5×5 lattice, but to provide this mode in a 9×9 lattice, first we have to disconnect the lattice nodes, then tune all lattice maps to a dynamic mode with λ as high as $\lambda = 0.24$, and finally, restore the connections.

Obviously, in many cases this method is inadmissible for practical applications, but we believe it might well take place in some natural systems with spatiotemporal chaos...

4.2. Increased connection depth

As is known [Dmitriev *et al.,* 1995, 1996, 1997], in the case of global (all to all) couplings of the elements of ensemble (1), there always is a range of the coupling strength that ensures global synchronization. Consequently, an increase of the neighborhood must sooner or later lead to global synchronization of the lattice of chaotic maps.

We considered the effect of the connection depth numerically in a 13×13 lattice. We used full templates, i.e., a current node was equally connected to all the nodes in its neighborhood. The results of calculations of the stability of synchronous mode are given in Fig. 5, where we can see how the effect of the topological term becomes more and more profound.

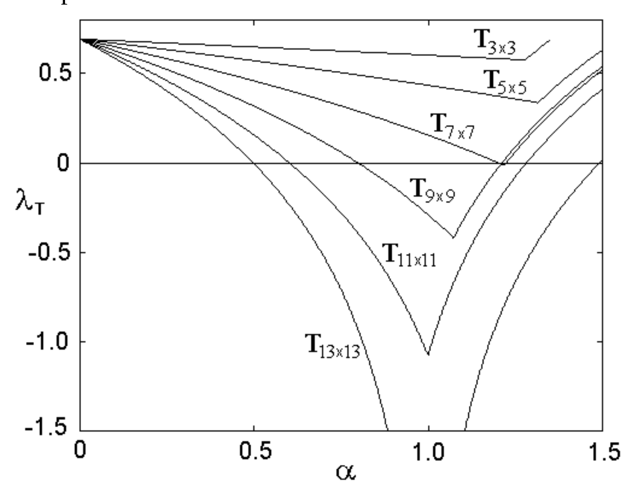

Fig. 5. Stability of the synchronous mode in 13×13 lattice at different dimensions of the local neighborhood. (The synchronous mode is stable if $\lambda_T < 0$).

Numerical investigations of large lattices of logistic maps show that the synchronous mode becomes possible when the linear size of the local neighborhood becomes equal to approximately half the size of the entire lattice. Thus, synchronization becomes possible, but the number of connections in the system dramatically increases, which ruins the very idea of local connections.

4.3. Nonlocal connections

Judging from the above results, one might make a conclusion that the stability of the synchronous mode in large lattices is determined by the number of connections in the systems. It might seem that as the number of connections decreases, the stability conditions get worse, and vice versa.

However, investigation of 1-D rings and 3-D bulk lattices of chaotic maps with full neighborhood templates, where the number of connections per node is less and, respectively, much more than in 2-D lattices, showed that the stability conditions in all these cases coincided. The plot in Fig. 4 with the linear dimension of the lattice depicted along the *Y*-axis equally describes 1-D, 2-D, and 3-D lattices.

This surprising result leads us to the following conclusion: not the number of connections but their length is crucial for the synchronous mode, i.e., to get the lattice synchronized, long-range connections are necessary.

To justify this statement, we investigated a lattice with a small number of nonlocal connections. We fixed the number of connections per node, and randomly set the pattern of connections which was the same for all lattice nodes. This looks as if the neighborhood template **T** were increased to the size of the entire lattice, and only few entries in the template were nonzero.

In Fig. 6, the topological term is shown for 16×16 lattice with 16 connections per node. The bunch of curves corresponds to 10 tests with random patterns of connections. Since the isolated logistic map Lyapunov exponent is $\lambda \approx 0.7$, only the neighborhoods corresponding to the curves that run below –0.7 level will synchronize the lattice.

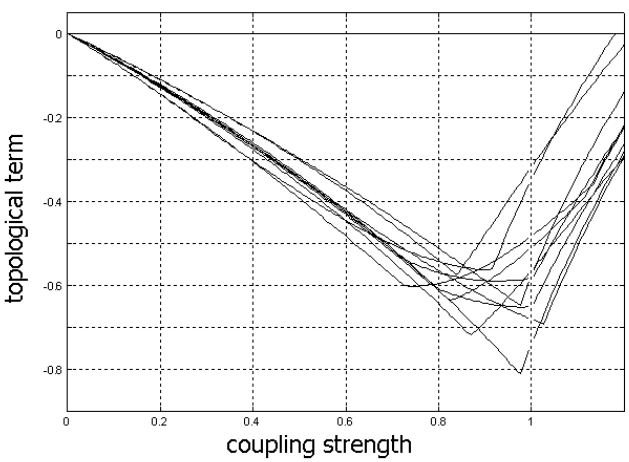

Fig. 6. Ten random neighborhoods.

Note, that the number of connections per node here is only 16, which is much less than in the case of wide local neighborhood discussed above. Indeed, to stabilize this 16×16 lattice using the method of increased connection depth, we must use at least 9×9 neighborhood template which gives 80 connections per node.

As can be seen, the topological term is very different in these tests: its minimum ranges from –0.56 to –0.81. Some connection patterns satisfy the stability condition and the others not, i.e., there are "good" and "bad" connection patterns. What is the best pattern (with the lowest minimum) is a special question.

The connections we discussed by far were static. However, if we change the connection pattern each time step, i.e., make connections dynamic, we may try to avoid the effect of "bad" patterns.

Eventually, we found that if the connection pattern is randomly changed each time the lattice is iterated, three connections per node are enough to get the entire lattice synchronized. This is an experimental result, which is valid for lattices as large as 100×100.

Thus, if we refuse the idea of only local connections, we can essentially decrease their number in large lattices, e.g., 80 *local* connections are necessary to synchronize a 16×16 lattice, but only 16 *static* or even 3 *dynamic* random connections per node can be enough.

4.4. Pacemaker-controlled lattices

As we have already found, global synchronization is impossible in large lattices of locally coupled chaotic elements (maps). This result is universal and holds for any type and dimensionality of the maps. So, in order to provide synchronization, we have to take some special efforts.

Let us add an element to the above lattice, the same chaotic element with the same system parameter, and endow it a special role of a pacemaker, i.e., connect this dedicated element to all other lattice elements unidirectionally, with no reverse action. Let β be the coupling strength. The dynamics of this independent element is described by the mapping equation f, while the behavior of the lattice elements is now described as follows

$$x_{ij}(n+1) = f\left(x_p b + (1-b)\left[\frac{a}{m}\sum_{s}^{m} x_{is} + x_{ij}(1-a)\right]\right), \quad (4)$$

where x_p is the pacemaker variable. If the pacemaker is disconnected from the lattice, i.e., $\beta = 0$, this relation is reduced to the above relation for the lattice of locally coupled chaotic elements (3).

As follows from the dynamics of coupled discrete-time systems [Dmitriev *et al.,* 1996, 1997], in the case of two identical unidirectionally coupled maps, the synchronous mode always exist, which may be proved directly, and the stability condition with respect to small perturbations in linear approximation can be expressed by the inequality

$$(1 - \beta) < \exp(-\lambda), \quad (5)$$

where λ is the maximum Lyapunov exponent of the first map trajectory. In the case of logistic maps with $\mu = 4$ and $\lambda = \ln 2$, the condition for the synchronous mode is $\beta \geq 0.5$. Experiments confirm this theoretical result. At $\beta \geq 0.5$ and any strength of internal coupling α, the global spatial synchronous mode is observed.

As follows from the theory [Dmitriev *et al.*, 1995], if we decrease β to β < 0.5 in the case of two maps, the synchronous mode is destroyed and can be observed no more.

Surprisingly, we found that in the case of β < 0.5 even in the uncoupled lattice a global synchronous mode is observed. All the lattice maps behave coherently, though they may not be coupled with each other at all (α = 0), i.e., an explicitly synchronous mode exists. However, this mode is different from the pacemaker mode. It is also chaotic (λ = 0.52÷0.69) and stable, and attracts the lattice maps from all initial conditions. Local connections do not change this mode, it exists at any α and β ∈ [0.332..., 0.5].

Obviously, the existence of this generalized synchronization [Abarbanel *et al*., 1996] [Kocarev & Parlitz, 1996] could have been explained if we could prove that the oscillation mode in the lattice maps induced by the pacemaker were stable with respect to small perturbations. To investigate the situation, consider two unidirectionally coupled logistic maps. Let us vary the coupling β in the range [0, 0.5] and investigate the stability of the motion induced in the second map with respect to small perturbations in this map (not necessarily with respect to small perturbations in the first map). In order to estimate the stability, we calculate the Lyapunov exponent of the second map trajectory as of an induced motion, i.e., as that of a nonautonomous system

$$\lambda = \ln\left(\lim_{N\to\infty}\left(\prod_{k=1}^{N}\left|\frac{\partial f\left(\beta x_1(k) + (1-\beta)x_2(k)\right)}{\partial x_2}\right|^{1/N}\right)\right) = \ln\left((1-\beta)\lim_{N\to\infty}\prod_{k=1}^{N}\left|\frac{\partial f\left(x_2(k)\right)}{\partial x_2}\right|^{1/N}\right). \quad (22).$$

The plot of the Lyapunov exponent as a function of the coupling β is presented in Fig. 7. One can distinguish three main modes of the second map motion related to different ranges of the parameter β. The range ***I*** = [0.5, 1.0] corresponds to the ordinary synchronous mode, and the curve of the Lyapunov exponent behaves as $\lambda^* = \ln(1-\beta) + \lambda$, where λ = ln2 is the Lyapunov exponent of the isolated map. The Lyapunov exponent is negative here, so this synchronous mode is stable.

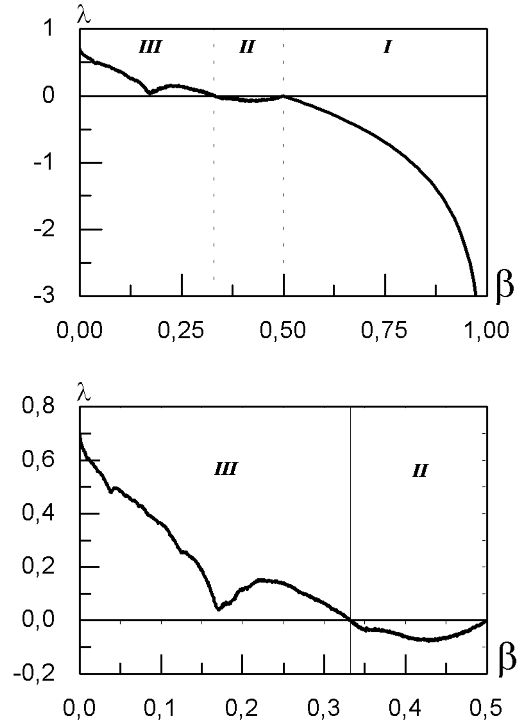

Fig. 7. Lyapunov coefficient of the trajectory induced in the second map.

As β decreases and enters the second range ***II*** = [0.332..., 0.5], the synchronous oscillations are destroyed, however the motion observed now in the second map does not become independent. The induced chaotic oscillations in the second map are stable, which follows from the negative Lyapunov exponent. Thus, a new attractor appears in the phase space of the entire dynamic system. This attractor does not lie along the main diagonal of the phase space (Fig. 8). We will call this mode "hidden" synchronization, because it is responsible for the synchronous motion of the chaotic lattice in the case when it is not evident. Within range ***II*** the form of the attractor remains the same.

Finally, the third type of the motion takes place at β ∈ ***III*** = [0, 0.332...]. The Lyapunov exponent of the induced trajectory becomes positive, and as β → 0, the system motion rapidly becomes independent.

The obtained results help us to explain the phenomena in a lattice composed of chaotic 1-D maps.

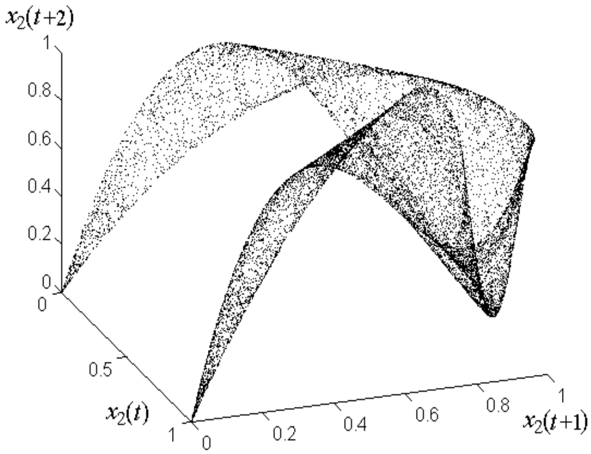

Fig. 8 Attractor of zone **II** reconstructed in a time-delay space.

In the motion of the lattice driven by the pacemaker, we can distinguish three different modes: the mode of total synchronization at $\beta \in [0.5, 1]$, the mode of spatially synchronous lattice observed at $\beta \in [0.332..., 0.5]$ and any α ("hidden" synchronization), and desynchronization mode at lower β.

We have found experimentally that the spatial synchronous mode survives in a narrow ribbon of $\beta \in [0.31\ldots, 0.332...]$ and nonzero α. The closer β to the boundary of ***II*** and ***III*** zones, the faster is the convergence to the global synchronous mode. Typical values of the convergence time are about 100 steps for the value of local connections α close to 1. As α is decreased and β lowered toward $\beta = 0.31\ldots$, the convergence time increases to tens of thousand steps. Below $\beta = 0.31\ldots$, the system becomes desynchronized, though near this value and at very weak local connections, intermittency between the spatial chaos and spatial synchronous mode is explicitly observed.

Conclusions

We found rigorous solution to the stability condition of the spatially homogeneous synchronous mode of lattices of chaotic maps with local connections and related it to the lattice size and the strength of local connections.

We found that the global synchronous mode is impossible in large lattices with local connections and proposed four mechanisms that can provide it:

— tuning of the dynamic mode of the lattice maps;
— large local neighborhoods;
— static and dynamic nonlocal connections;
— introduction of a pacemaker.

In a model of a pacemaker-controlled lattice, we have found a spatial synchronous mode induced by the external pacemaker, different from the pacemaker mode. On the edge of spatially synchronous and desynchronized mode regions, the presence of local connections preserved the stability of the spatially homogeneous chaotic mode.

References

Abarbanel, H. D. I., Rulkov, N. F. & Sushchik, M. M. [1996] "Generalized synchronization of chaos: The auxiliary system approach", *Phys. Review E* **53**(5), 4528–4535.

Chua, L. O. & Yang, L. [1988] "Cellular neural networks: Theory", *IEEE Trans. Circuits and Systems — I: Fundamental Theory and Applications* **35**(10), 1257–1272.

Dmitriev, A. S., Shirokov, M. E. & Starkov, S. O. [1995] "Chaotic synchronization of ensembles of locally and globally coupled discrete-time dynamical systems. Rigorous results and computer simulation," *Proc. Workshop NDES-95*, Dublin, Ireland, 287–290.

Dmitriev, A. S., Starkov, S. O. & Shirokov, M. E. [1996] "Synchronization of ensembles of coupled maps," *Izv. Vyssh. Uchebn. Zaved. — Prikladnaya Nelineinaya Dinamika* **4**, Nos. 4–5, 40–58, (in Russian).

Dmitriev, A. S., Shirokov, M. E. & Starkov, S. O. [1997] "Chaotic synchronization in ensembles of coupled maps", *IEEE Transactions on Circuits and Systems-I* **44**, 918–926.

Gell-Mann, M. [1995] "What is complexity?", *Complexity* **1**(1), 16–19.

Gutowitz, H. [1995/1996] "Cellular automata and the sciences of complexity", *Complexity* **1**(5), 16–22.

Heagy, J. F., Carrol, T. L. & Pecora, L. [1994] "Experimental and numerical evidence for riddled basins in coupled chaotic systems", *Phys. Ref. Lett.* **73**, 3528–3531.

Kaneko, K. [1989] "Spatiotemporal chaos in one- and two-dimensional coupled map lattices," *Physica D* **37**, 60–82 (and references therein).

Kocarev, L. & Parlitz, U. [1996], *Phys. Rev. Lett.* **76**(11), 1816–1819.

Langton, C. G. [1986] "Studying artificial life with cellular automata", *Physica D* **22**, 120–149.

Maystrenko, Y & Kapitaniak, T. [1996] "Different types of chaotic synchronization in two coupled piecewise linear maps", *Phys. Review E* **54**, 3285–3292.