LIMIT EFFICIENCY OF CHAOTIC SIGNAL CLEANING OFF NOISE

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ABSTRACT

The relationship of the problem of chaotic signals cleaning off noise and information properties of signals is shown. Some methods for chaotic signals cleaning off are considered. An efficiency of these methods is evaluated by computer simulation.

1. INTRODUCTION

The problem of cleaning (filtering) chaotic signals off external noise is of great interest for many applications, it deserves intent attention from the theoretical point of view. In the simplest form, it could be stated as follows. Let there be a chaos source (*CS*) sending the signal $x(k)$ into a communication channel where a noise $w(k)$ is added to the signal. Then, there is a device called a chaos receiver (*CR*). The mixture of the chaotic signal and noise $z(k)$ is fed to the *CR* input, and a chaotic signal as close as possible to the *CS* output signal $\hat{x}(k)$ must appear at the CR output (Fig. 1).

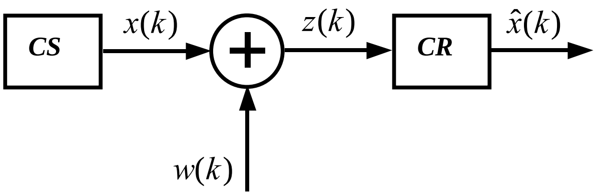

Fig.1. The scheme of chaos cleaning off noise : *CS* - chaos source, CR - chaos receiver, $x(k)$ - chaotic signal; $w(k)$ - noise, $z(k)=x(k)+w(k)$; $\hat{x}$ (k) - estimation of the chaotic signal.

In this statement, the problem of chaotic signal cleaning is a version of the classical problem of a signal transmission over a communication channel with noise with its further estimation [1,2], where the chaotic signal plays the role of an information signal.

On the other hand, it has some common features with the problem of obtaining synchronous chaotic response or the problem of synchronization of CS with CR [3] when the signal in the channel is disturbed by interference.

Finally, this problem can be discussed as a problem of obtaining a copy of the signal generated by CS at the output of the nonlinear device, the chaos receiver, as precise as possible.

The problem of cleaning chaotic signals was analyzed in a few publications by means of different approaches that did not take into account the information properties of the chaotic signal. The significance of the information aspect of the problem was noted in [4,5]. However, the relation between the cleaning efficiency and information properties of chaotic signals was not considered in detail in these publications.

The aim of this paper is to analyze the information aspects of the chaotic signal cleaning (filtering), to find out the limit potential efficiency of the chaotic signal cleaning and to investigate concrete cleaning algorithms from the view point of achieving the maximum efficiency.

2. INFORMATION PROPERTIES OF CHAOTIC SIGNALS

Consider the production and disappearance of information in nonlinear systems with complicated behavior on example of 1-D maps of the unit interval into itself

$$x(k+1) = f(x(k), \mu) \tag{1}$$

Average increase of information $\bar{\lambda}$ on the whole interval is expressed by the integral:

$$\bar{\lambda} \equiv \Delta H_{av} = \int_0^1 P(x) \log_2 \left|\frac{dy}{dx}\right| dx, \tag{2}$$

where $P(x)$ is the probabililty density.

The value of $\bar{\lambda}$, which is the mean value of the information produced at one step of the map iteration, can be determined even if $P(x)$ is unknown. To do this, we must iterate the map starting from some initial point and calculate the mean value of the slope logarithm

$$\bar{\lambda} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log_2 \left|\frac{dy}{dx}\right| \tag{3}$$

The obtained sum for *ergodic* map is assumed to be weighted with the probability density $P(x)$ due to the very process of iteration.

The information $\bar{\lambda}$ is easily calculated for some simple maps, for instance, for the family of "tent" maps, described by equation (1) with the right hand side

$$f(x,\mu) = \begin{cases} x/\mu, для - x < \mu. \\ \dfrac{1-x}{1-\mu}, для - x > \mu. \end{cases} \tag{4}$$

These maps have constant probability density $P(x)=1$. The expression for $\bar{\lambda}$ has the following form:

$$\bar{\lambda} = -[\mu \log_2 \mu + (1-\mu)\log_2(1-\mu)]. \tag{5}$$

The maximum rate of information production in (4) is equal to one bit per iterate and is achieved in the symmetrical map with μ=0. 5.

As the second example, consider the logistic parabola map $x(k+1) = \mu x(k)\,(1-x(k))$

In the case of $\mu = 4$, this map is ergodic and $\overline{\lambda}$ is equal to one bit per iterate. The slope of map $y = \mu x(1-x)$ is not greater than one in all the points. So, the absence of stable periodical points can not be quarantined. It leads to complicated relations (for μ<4) between chaotic and stable periodical modes.

Both quantitative estimates of the mean rate of the information production and the fluctuations of this rate in time domain are of interest. Indeed, the systems having the some mean rate of the information production can have different production rates at some iterates (at different moments). The differences can be either insignificant (in the limit case, no differences) or very important. This difference, as will be shown below, plays an important role in the cleaning of chaotic signals off noise.

Consider the information production fluctuation in the discussed examples.

The slope of the Bernoulli shift map is constant and the mean rate of the information production coincides with the rate of its production at any iterate. Distribution density for this rate is a δ-function.

The information production of the asymmetrical tent map at one iterate can take two values:

$$\overline{\lambda} = \begin{cases} -\log_2 \mu, & x < \mu \\ -\log_2 (1-\mu), & x > \mu \end{cases} \tag{6}$$

Since the distribution probability density of the variable is constant within the interval (0,1), the number of iterates with each of the two values of λ^i is proportional to μ and $1-\mu$, respectively. Therefore, the distribution probability density has the following form

$$\begin{aligned} P(I) &= \mu\delta(I + \log_2 \mu) + \\ &+ (1-\mu)\delta(I + \log_2 (1-\mu)) \end{aligned} \tag{7}$$

For the logistic map, the information production distribution density varies strongly in depending on the value of parameter μ..

3. RESTRICTIONS IMPOSED BY THE INFORMATION THEORY ON THE CLEANING OF CHAOTIC SIGNALS

Thus, the dynamical system generating chaos is a specific source of information messages with the mean rate of information production $\overline{\lambda}$.

In accordance with the Shannon theorem, to transmit an information volume $\overline{\lambda}$ in a unit time the minimum channel capacity C must satisfy the following relation:

$$C > \overline{\lambda}. \tag{8}$$

Relation (8) does not give a procedure for cleaning chaotic signals off noise. The sense of (8) is that the "communication channel" must possess some minimum channel capacity to clean chaotic signals. In other words, the noise level have not to exceed some top permissible level.

The quantitative analysis is based on the theorem about the capacity channel with noise [2]. According to the theorem, the capacity of the channel with frequency band width W and white thermal noise with the power N and average transmitted signal power S is equal to

$$C = W \log_2 \frac{S+N}{N}. \tag{9}$$

That is, by means of encoding, we can transmit binary digits with the rate of $W \log_2 \frac{S+N}{N}$ bps and a as small as necessary error rate.

According to relation (9), judging from the value $\overline{\lambda}$ of the information produced by the driving system and from the channel bandwidth, we can determine the maximum admissible noise level at which the synchronization is still possible. The "expense" for the possibility of synchronization in the presence of noise is a special organization of the information transmitted through the channel and a certain time delay of the processes in the driven system with respect to those in the driving system. Note also, that by practical accomplishment of precise synchronization in the presence of noise a certain redundancy of the channel capacity is necessary in respect to the value given by relation (5). Otherwise, the delay value will tend to infinity.

4. CLEANING OF CHAOTIC SIGNALS FROM NOISE

Let us demonstrate a practical possibility of cleaning chaotic signals off noise taking into account their information properties.

The first approach. Let a chaotic dynamic system be Bernoulli shift map

$$f(x,\mu) = (x/\mu)\,\mathrm{mod}1. \tag{10}$$

For μ =2, in binary representation, this map moves mantissa to the left by one position.

Let us divide the interval [0,1] into two parts [0,1/2) and [1/2,1] and assign the symbolic variable $S(k)$ to 0 if $x(k) < 1/2$ and to 1 if $x(k) \geq 1/2$. This means, that $S(k)= a_k$ and the symbolic sequence $S(1),\ldots S(n)$ contains all information about $x(1)$ with the accuracy within n binary digits.

The mixture of the chaotic and noise signals $z(k)=x(k)+w(k)$ is fed to the driven system. Let there be $< x^2(k) > \quad >> \delta^2 =< w^2(k) >$, then for most iterates the integer parts of doubled values $z(k)$ and $x(k)$ will be equal to

$$int(2x(k))=int(2z(k))=S(k). \qquad (11)$$

Thus, $x(1)$ can be recovered as accurate as n binary digits from the first n samples. In other words, at the driven system output, we can obtain the signal $\hat{x}(1)$ coinciding with the driving system signal with a good accuracy. Of course, relation (11) brakes from time to time but the level of noise the signal at the driven system output is considerably lower as compared to that in the signal in the communication channel.

Note the two circumstances.

First, in the driven system, we use only very rough information about the chaotic samples namely only one information bit (0 or 1). This is just the amount of information produced by the CS by one iterate. This "roughness" allows us to increase the probability of obtaining the correct value for the sample estimate.

Second, $x(1)$ can be recovered with accuracy n-bit only after a series of n samples $z(1),\ldots z(n)$ is recovered. That is, $x(k)$ is recovered with a delay in respect to the reception of $z(k)$ sample in the driven system. This delay is the more, the stricter the recovery accuracy requirement, the longer the delay.

The procedure of signal cleaning can be modified as follows.

The second approach. Let us introduce a map inverse to the Bernoulli shift map. Note, that it is two-valued contracting map (contraction value is equal to 2).

Fix a time moment k' and a natural number $n\geq 1$. To get an estimate of $\hat{x}(k')$, we use the following algorithm. Consider a piece $\{z(k)\}_{k=k'}^{k=k'+n}$ of the observed trajectory. The idea of the clearance is that map (1) is strictly contracting at backward iterations, i.e., iterating sample $z(k'+n)$ backward n times allows us to approach the true sample $x(k')$ close enough. For Bernoulli shift map, the contraction at backward iteration is equal to 2, and deviations of the estimate from the true value is equal to $w(k'+n)2^{-n}$.

Direct accomplishment of this approach is impossible, since in map (10) two argument values correspond to a single function value. This problem can be solved using information present in the interval $\{z(k)\}_{k=k'}^{k=k'+n}$ of the observed trajectory in order to select to one of the two branches of the inverse map.

To take the correct branch of the inverse map at each backward iterate we can use the sequence of $S(k)$. If $S(k\text{-}1)=0$, we take the lower branch of the inverse map, and the upper branch for $S(k\text{-}1)=1$. As well as in the first approach, the estimation accuracy of $x(k)$ sample value also increases by a factor of two by one step of the map iteration. However, in the case of a small noise with $\delta^2 << 1$, the initial uncertainty is equal to 1/2 for the first approach and δ for the second. Therefore, for the same estimation accuracy the second approach requires number of iterates less time (delay).

The third approach. According to approach to chaotic signal cleaning for the Bernoulli shift map, the branch for the backward iterate of a current sample $z(k)$ is chosen using the knowledge the previous sample $z(k\text{-}1)$ instead of the symbolic sequence element $S(k\text{-}1)$. The idea of the approach is as follows.

At the first step, we choose one of the two pre-images of point $z(k'+n)$, the one that is the nearest to the sample $z(k'+n\text{-}1)$. We denote as $y(k'+n\text{-}1)$. At the second step, of the two pre-images of the sample $y(k'+n\text{-}1)$ we take the nearest to $z(k'+n\text{-}2)$ and denote it as $y(k'+n\text{-}2)$. The process is repeated until we get the series of samples $\{y(k)\}_{k=k'}^{k=k'+n-1}$ These samples are the "true" trajectory and the value $y(k')$ is chosen as an estimate of $\hat{x}(k')$ for $x(k')$.

The discussed procedures of chaotic signal cleaning off noise for map (10) allows us to generalize the results to other 1-D maps (with a finite number of inverse map branches). In general, the cleaning quality for each piece of the trajectory will be determined by the local Lyapunov exponent (local rate of information production) for this segment.

"Backward analysis depth" n is a free parameter of the procedure. If we choose the "true" trajectory correctly, the clearance quality increases exponentially with n (as 2^n, in the case of Bernoulli shift map (10)).

However, an increase of n can result in increased probability of false choice of the "true" trajectory, which it leads to decrease of the clearance quality.

In order to estimate the efficiency of the discussed method, we simulate the cleaning of chaotic signals generated by Bernoulli shift map (10) with Gaussian noise in the channel.

In the simulation using the second and third methods, the initial error was equal to δ, while for the first approach it was equal to 1. Each backward iterate decreases the noise level by 6 dB. Hence, to get comparable results, the number backward iteration steps in the first approach must be greater than in the second and third approaches by

$$\Delta n = [20\lg(1/\delta)]/6 \approx 3.3\lg(1/\delta).$$

The numerical results are shown in Fig. 2. Analysis of the plots in Fig. 2 shows that the third method (curve 3) is the most efficient and that it begins functioning at higher noise levels.

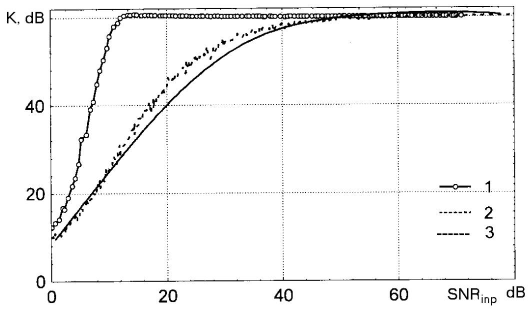

Fig.2. The comparison of the efficiency of cleaning methods (1-3) for Bernoulli shift map (n=10, Gaussian noise): cleaning coefficient as a function of noise level in the channel.

The first and the second methods (curves 2 and 3) are nearly equivalent at a proper number of steps.

5. FACTORS INFLUENCING THE EFFICIENCY OF THE CLEANING ALGORITHM

One can expect that the cleaning efficiency depends on the kind of the map generating the chaotic signal. Indeed, consider tent map (4) with μ=0.5.

This map as well as the map of Bernoulli shift (10) has constant invariant measure on [0,1] and fixed slopes of the map function segments, hence, a constant value of the local Lyapunov exponent along the trajectory.

In Fig. 3, logarithm of the cleaning efficiency *K* as a function of the Gaussian noise in the channel is plotted for the chaotic signal generated by the symmetric tent map (curve 2). A comparison with the similar plot for the signal produced by the Bernoulli shift map (curve 1) demonstrates significant degradation of the algorithm efficiency.

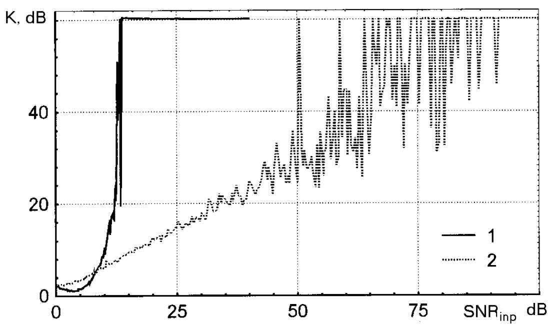

Fig.3. The cleaning coefficient as a function of Gaussian noise level in the channel for signals generated by the maps (10), (4) (n=10, μ =0.5).

Let us discuss the reasons for this effect.

The distance between the branches of the map inverse to the Bernoulli shift is always equal to 0.5. In contrast, the distance between the branches of the map inverse to the tent map is a function of variable x and is zero in the point x=1.0.

This means that for the map inverse to the Bernoulli shift the branch is chosen correctly if the error is less than 0.5. For the map inverse to the tent map no guaranties could be given. The algorithm can function wrong in one of the series points at any noise level. Of course, the lower the noise level in the channel, the less the probability of the error.

At very low noise, for both maps the method gives the theoretical value of the cleaning efficiency $K = 20\,\lg(2^n)$.

The obtained results indicate that the geometry of the map has a strong effect on the efficiency of the chaotic signal cleaning procedure even if such characteristics as the invariant measure and the local Lyapunov exponent behavior remain constant. We call the effect of the map geometry on the quality of cleaning the geometric factor.

6. CONCLUSIONS

In this paper, we have shown the existence of fundamental relations that determine the possibility of cleaning chaotic signals off noise. The sense of these relations is that the chaotic oscillations contain information. Consequently, in order to exactly reproduce the chaotic oscillations in the receiver a certain minimum of necessary information must not be lost. This minimum is determined (at least for one-dimensional maps as chaotic sources) by the degree of the signal chaoticity, which is equivalent to the average information containing in each sample, and by the noise level.

A principle feature of the process of cleaning chaotic signals off noise is its exponential efficiency: SNR (in dB) increases linearly (in logarithmic scale) with increasing number of iterates, while for regular signals SNR (in dB) increases only as a logarithm of the number of iterates.

7. REFERENCES

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