INFORMATION VIEWPOINT ON CHAOTIC SYNCHRONIZATION

Alexander S. Dmitriev
Institute of Radio Engineering and Electronics, Russian Academy of Sciences,
Mokhovaya 11, Moscow, 103907, Russia
e-mail: chaos@mail.cplire.ru

Gennady A. Kassian and Alexander D. Khilinsky
Moscow Institute of Physics and Technology,
Dolgoprudny, Moscow region, 141700, Russia

Abstract

The phenomenon of chaotic synchronization is investigated from information viewpoint. Synchronization of a "receiver" of chaos with a chaotic "source" is discussed as an ability of the receiver to recover a copy of the chaotic signal generated by the source. Here we show that in terms of the information approach the condition for the chaotic synchronization is not the level of the physical effect of one system on the other, but the transmission of a necessary information about the chaotic process and, hence, the information carrying capacity of the "communication channel" between the source and the receiver.

1. Introduction

The phenomenon of chaotic synchronization of two systems with uni-directional coupling implies, as a rule, a direct effect of a physical process x_n in one system, the chaotic source (*CS*) or drive, on the other system, the receiver of chaos (*CR*) or response, in the presence of, in general, noise w_n in the channel.

The case when two systems are not synchronized directly through the physical signals generated by them, but using a number of signal transformations is rather typical also for the systems with regular processes synchronized.

So, circuit is more realistic (Fig. 1). Here, along with the drive and response systems, a transformer (*Tr*) of the physical signal from one form to another, an inverse transformer (Tr^{-1}) and a channel that carries the signal in a new form y_n from the transformer to the inverse transformer are presented.

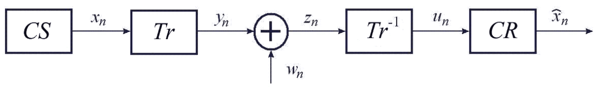


Fig. 1. Synchronization with signal transformation. *CS* - chaos source; *CR* - chaos receiver; *Tr* - transformer; Tr^{-1} - inverse transformer.

The physical nature of the signal in the channel, after the transformer, may be different from that of the signal at the output of the drive system. For example, the drive and response systems can be mechanical, with an optical signal in the channel. In general, a white noise w_n is added to the physical signal y_n in the channel. The signal $z_n = y_n + w_n$ is then fed to the input of the inverse transformer.

The powers of the signals at the outputs of the drive system and the transformer can also be different. However, at the response system input we must again have a physical signal u_n characteristic of the synchronized systems. It is recovered at the reception site from the signal that comes from the channel.

Physical signal u_n is recovered at the reception site from the signal that comes from the channel. This means that we substitute the direct physical effect of one system on the other by its imitation, by reproducing the necessary signal at the response system input, i.e., in reality we change the physical effect of the systems by an information about this effect. Or, in other words, replace the physical interaction by the information interaction. The above discussion permits us to introduce the following definition of the chaotic synchronization for drive–response systems [1].

Definition. Let there be a drive and a response systems. We say that the response system is synchronized with the drive system if the signal at the output of the response system is a copy of the signal at the output of the drive system irrespective of whether they interact directly through the physical signal generated by the drive system, through the signal transformed to some other form and then transformed back to the initial signal, or with a help of information delivered from the drive to the response system.

2. Information Content of Chaotic Signals

Before we proceed to an analysis of the information effect of the drive system on the response one, consider the production and disappearance of information [2]-[4] in a dynamic system of a mapping of the unit interval into itself

$$x_{n+1} = f(x_n) \tag{1}$$

where $f(\cdot)$ is a nonlinear function.

Initial value x_0 can always be known only with a certain accuracy e, which means $h = -\log_2(e)$ bits of information. An increase of information in point x of map $y = f(x)$ is determined by the slope of the function curve in this point

$$\Delta h = \log_2\left|\frac{dy}{dx}\right| \tag{2}$$

For example, in the case of the map of Bernoulli shift

$$x_{n+1} = (2x_n)\mathrm{mod}1, \tag{3}$$

the change of the information about the point location is

$$\Delta h = [\log_2(2e) - \log_2(e)] = \log_2 2 = 1 \text{ bit}$$

After $n \approx -\log(e)$ map iterates, the initial uncertainty e leads to uncertainty within the entire interval (0,1) and the data about the initial point location is lost.

Average value $\overline{l}$ of information produced by one iterate is expressed through integral (2) weighted with the probability density $P(x)$

$$\overline{l} \equiv \Delta H_{mean} = \int_0^1 P(x)\log_2\left|\frac{dy}{dx}\right|dx\,. \tag{4}$$

However, we can determine $\overline{l}$ even if $P(x)$ is unknown. To do this, we iterate the map starting from some initial point and calculate the mean value of the slope logarithm

$$\overline{l} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log_2\left|\frac{dy}{dx}\right|_{x=x_i}. \tag{5}$$

As is assumed in (5), for the ergodic map the obtained sum is weighted with the probability density $P(x)$ due to the very iteration process.

The Lyapunov exponent can be treated as the information production rate expressed in base-e units. In order to transform it to bits-per-iterate, multiply λ by $\log_2 e$

$$\overline{l} = l\log_2 e \tag{6}$$

3. Restrictions Imposed by Information Theory on Synchronization of Chaotic Systems

Specific features of the chaotic source of messages are the finite rate of information production and the continuous set of the signal values.

According to the Shannon theorem [5], in order to transmit information volume I per unit time, the minimum information carrying capacity C of the channel must satisfy the relation $C \geq I$.

Consider a "communication system" composed of a drive and a response chaotic systems coupled uni-directionally

$$x_{n+1} = f(x_n) \tag{7 a}$$

$$y_{n+1} = f(y_n + \alpha(x_n - y_n)). \tag{7 b}$$

where a is the coefficient of coupling between the drive and response systems and $(1-a)$ is the feedback coefficient in the response system.

Necessary conditions for synchronization of the drive and response systems (7) can be expressed analytically [2] as

$$l\perp = \ln|1 - a| + l < 0, \tag{8}$$

where $l\perp$ is the Lyapunov exponent in the direction transverse to the synchronization attractor plane. Synchronization occurs only when the value of $l\perp$ becomes less than zero.

***Corollary* 1**. *The "noiseless channel" corresponds to the case of the system with disconnected feedback loop, i.e.,* $a = 1$.

The value of $l_\perp = 0$ (see (8)) determines the boundary of synchronization stability

$$l = -\ln|1 - a|. \tag{9}$$

***Corollary* 2.** *The case of no external noise and no feedback in the receiver corresponds to the communication channel with* $C = \infty$. *Synchronization is possible at any high rate of information production by the chaotic system.*

The above analysis shows that for synchronization of the drive and response systems in the absence of noise it is sufficient to have a "channel" with the information carrying capacity

$$C > \bar{l}\ . \tag{10}$$

The theorem for the information carrying capacity of the channel with noise [5] gives the basis for quantitative analysis. According to this theorem, the capacity of a channel with the frequency bandwidth W is equal to

$$C = W \log_2 \frac{P+N}{N}. \tag{11}$$

This means that using a special encoding one can transmit binary digits with the rate of $W \log_2 \frac{P+N}{N}$ bps with no matter how small error rate.

4. Synchronization of Chaotic Systems in the Presence of Noise in the Channel

Treating synchronization as an information process, let us discuss two synchronization methods. We shall use the Bernoulli shift map and the tent map as chaotic sources.

Consider a map

$$x_n = f^{-1}(x_{n+1}) \tag{12}$$

inverse to map (1). Let the symbolic sequence $X_1, \ldots X_n, \ldots X_{n+k}\ldots$ be fed to the receiver of chaos instead of the original chaotic sequence x_n. A term X_{n+k} of the symbolic sequence defines the value of the chaotic sample as accurately as semi-interval [0, 1/2) or district [1/2, 1] for $X_{n+k} = 0$ and $X_{n+k} = 1$, respectively. Let us apply map (12) to this semi-interval or interval. As a result, we obtain two segments, each of the size of 1/4. Of the two, we then take the segment corresponding to element X_{n+k-1} of the symbolic sequence. For example, if $X_{n+k-1} = 1$, we take the segment corresponding to the upper branch of map (12).

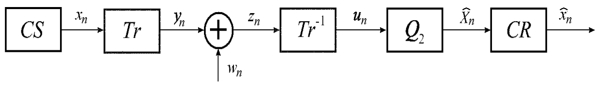


Fig. 2. First synchronization method using symbolic sequences. *CS* - chaos source; *CR* - chaos receiver; *Tr* - transformer; Tr^{-1} - inverse transformer; Q_2 - quantizer.

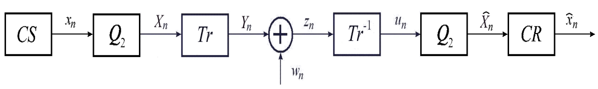
Thus, we obtain an estimate of x_{n+k-1}, now as accurate as a length-1/4 interval. Repeating the procedure by k iterates, we obtain an estimate of x_n within 2^{-k-1} accuracy.

The methods for synchronization in the presence of noise in the communication channel discussed below differ by the circuit forming the symbolic sequence.

In the first method (Fig. 2) the drive system generates a signal in the range [0, 1]. Then the signal is transformed according to the expression

$$y_n = 2(x_n - ½) \tag{13}$$

and sent through the communication channel where noise w_n is added.



.

Fig. 3. Second synchronization method using symbolic sequences. *CS* - chaos source; *CR* - chaos receiver; *Tr* - transformer; Tr^{-1} - inverse transformer; Q_2 - quantizer.

In the second method (Fig. 3), signal x_n from the chaotic source output is fed to a level-2 quantizer (Q_2), where it is transformed into an element of symbolic sequence $X_n = 0$ if $x_n < 1/2$ or $X_n = 1$ if $x_n > 1/2$, which in turn is transformed into a physical signal $Y_n = -1$ if $X_n = 0$ or $Y_n = 1$ if $X_n = 1$. The physical signal is sent to the channel where a noise w_n is added

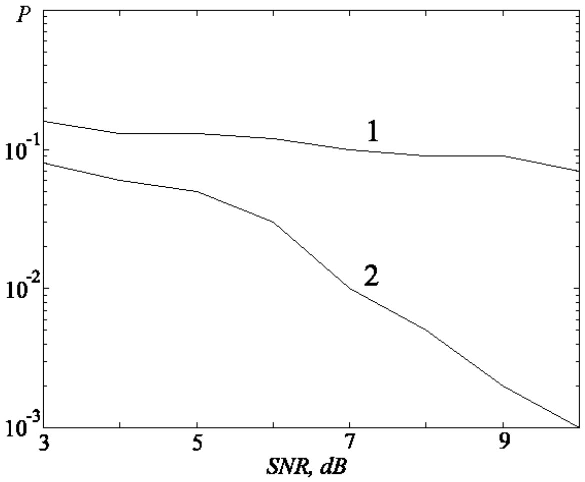


Fig. 4. Bit-to-error ratio (BER) for the case of Bernoulli shift map. 1 - first method; 2 - second method.

A priori, the second method seems to be less sensitive to errors than the first one.

On the other hand, in the case of quantizing the signal by two levels at the output of the chaotic source, a signal with well spread values "–1" and "+1" is sent to the channel and a much greater noise level is necessary in the channel to push the signal from –1 to +1 or vice versa, that makes the errors in the receiver quantizer less probable.

In Fig. 4-5, the calculation results for the probability of single errors of the discussed synchronization methods are presented (the first method – curve 1, the second method – curve 2) using Bernoulli shift (Fig. 4) and Tent map (Fig. 5) as chaotic sources, which confirm these conclusions.

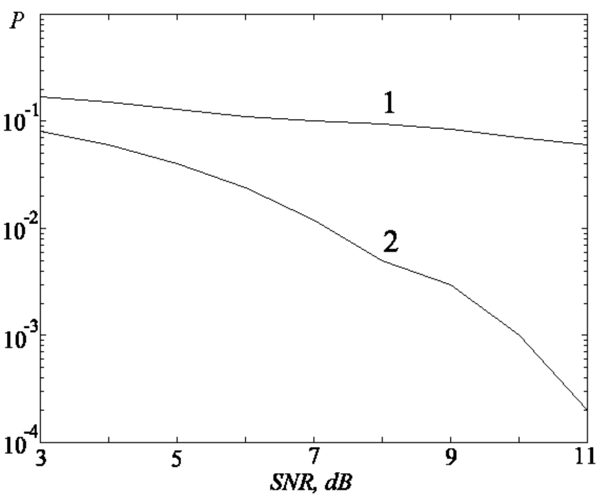


Fig. 4. Bit to error ratio (BER) for the case of Tent map. 1 - first method; 2 - second method.

Thus, effective methods for transmitting information generated by the map through a noisy channel essentially decrease the synchronization sensitivity in respect to noise and allow us to obtain synchronization of chaotic systems at a noise level close to the theoretical limit.

5. Conclusion

There are fundamental restrictions, imposed by the information theory, on the possibility of chaotic signal synchronization in the presence of noise. The reason for these restrictions is that chaotic oscillations are the information-bearing signals. Hence, in order to reproduce them exactly in the receiver, a certain minimum of information must be delivered. This minimum is determined (at least, for such chaotic sources as one-dimensional maps) by the degree of the signal chaoticity, which is equivalent to the average information that is contained in each sample, and by the noise level.

We also applied our theoretical results to the case of synchronization of weakly chaotic systems. The robustness of the developed method of chaotic synchronization is close to limit theoretical value.

Acknowledgments

The authors are grateful to Yu.Andryev, L.Kuzmin, A.Panas and S.Starkov for fruitful discussions and help in preparing of the manuscript.

The work is supported in part by the Russian Foundation for Basic Investigations (Grants No. 97–01–00800 and No. 99-02-18315).

References

[1]. A.S. Dmitriev, "Chaotic Synchronization as Infor mation Process," *Izvestiya Vuzov Radiofizika*, vol. 41, no. 12, pp. 1497-1509, 1998 (Russian).

[2]. R. Shaw, "Strange Attractors, Chaotic Behavior and Information Flow," *Naturforsch*, vol. 36A, pp. 80-112, 1981.

[3]. H.D.I. Abarbanel, *Analysis of Observed Chaotic Data*, Springer-Verlag, New York, Berlin, 1996.

[4]. A.S. Dmitriev, S.O. Starkov, "Message Transmis-sion Using Chaos and Classical Information Theory," *Uspehi Sovremennoy radioelektroniki*, no. 11, pp. 4-32, 1998 (Russian).

[5]. C.E. Shannon, [1948] "A Mathematical Theory of Communication," *The Bell System Technical J.*, vol. 27, pp. 379-423, pp. 623-656, 1948.

[6]. A.S. Dmitriev, M.E. Shirokov, S.O. Starkov, "Chaotic Synchronization in Ensembles of Coupled Maps," *IEEE Trans. Circuits Syst*., vol. 44, no. 10, pp. 918-926, 1997.