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EDGE OF SPATIOTEMPORAL CHAOS IN CELLULAR NONLINEAR NETWORKS

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ABSTRACT: *In this report, we investigate phenomena on the edge of spatially uniform chaotic mode and spatial temporal chaos in a lattice of chaotic 1-D maps with only local connections. It is shown that in autonomous lattice with local connections, spatially uniform chaotic mode cannot exist if the Lyapunov exponent l of the isolated chaotic map is greater than some critical value $l_{cr} > 0$. We proposed a model of a lattice with a pacemaker and found a spatially uniform mode synchronous with the pacemaker, as well as a spatially uniform mode different from the pacemaker mode.*

1. Introduction

A number of arguments indicates that along with the three well-studied types of motion of dynamic systems — stationary state, stable periodic and quasi-periodic oscillations, and chaos — there is a special type of the system behavior, i.e., the behavior at the boundary between the regular motion and chaos. As was noticed, processes similar to evolution or information processing can take place at this boundary, as is often called, on the "edge of chaos".

Here are some remarkable examples of the systems demonstrating the behavior on the edge of chaos: cellular automata with corresponding rules [1], self-organized criticality [2], the Earth as a natural system with the main processes taking place on the boundary between the globe and the space.

In contrast to the dynamic chaos, the discussed phenomenon, denoted by the term "complexity", is characteristic of the systems composed, as a rule, of many interacting elements. Often, such systems not only demonstrate the fourth type of motion, but also have adaptive features that draw the system to this edge. One may treat this adaptation as abrupt simplification of the system dynamics with respect to the "true" multidimensional chaos of the system elements in the absence of interaction.

What is the mechanism of the adaptation in coupled dynamic systems?

An attractive model of the edge of chaos is a generalized synchronization. Indeed, synchronization essentially simplifies the system dynamics by means of decreasing the attractor dimension. It is directly determined by the degree of the system couplings: the adaptive mechanism is turned on only in the presence of sufficiently strong coupling. As follows from investigations of the fine structure of chaotic synchronization attractors, perturbations can lead to on-off intermittency, or to local or global mode changes.

On-off intermittency, in turn, is a form of avalanches (breakdowns) characteristic of the systems on the edge of chaos.

In this report, we investigate the problem of synchronization of chaotic CNNs (here, chaotic cellular nonlinear networks) composed of chaotic maps and investigate its features as a system living on the edge of chaos.

2. Chaotic Cellular Nonlinear Network Model

The concept of an ensemble of chaotic elements that come to order through local interactions was investigated in a model composed of locally coupled 1-D maps. The dynamic system that we investigate here is a 2-D $M\times N$ lattice with the logistic maps as its nodes. All logistic maps $x(t+1) = f(x(t)) = \mu x(1-x)$ have the same parameter μ set at $\mu = 4$ (t is discrete time). The nodes x_{ij} of the lattice are connected to the nearest neighbors, so each map is coupled to, e.g., 8 nearest maps. The connection is described by the following relation

$$x_{ij}(t+1) = f\left((1-\alpha)x_{ij} + \frac{\alpha}{m}\sum_{s}^{m} x_{is}\right), \tag{1}$$

where x_{is} are the m nearest neighbors of x_{ij}. So, the dynamics of each node map is determined by its own dynamics and by an averaged effect of its neighbors. The boundary conditions are periodic: $x_{(M+1)j} = x_{1j}$ and $x_{i(N+1)} = x_{i1}$, i.e., we investigate a lattice on torus.

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The neighborhood is determined by a template **T** of the kind of $\mathbf{T}=\begin{pmatrix}1&1&1\\1&0&1\\1&1&1\end{pmatrix}$, where 0 corresponds to node x_{ij}, and the nonzero elements to those neighboring nodes that act on x_{ij} node. This template corresponds to neighborhood 3×3 (neighborhood of radius 1), however we also tried larger neighborhoods.. More complicated templates can be used in this model, e.g., with noninteger elements.

The object of this study was an appearance of order in the network composed of chaotic elements, so the parameter μ of all the maps was set constant and equal to $\mu = 4$, hence the isolated maps were in the state of the most developed chaos, i.e., characterized by the largest Lyapunov exponent and the maximum volume of the chaotic attractor in the phase space. Investigation of the system dynamics as a function of the parameter μ was beyond the scope of our interests.

The model was calculated and visualized in MATLAB environment. We studied the system behavior by means of iterating it starting with random initial conditions of the lattice nodes. The node state x_{ij}, a value in the range [0, 1], was depicted by a color from gray-scale palette, from black (0) to white (1).

3. Spatial Synchronization

We investigated the system behavior as a function of the local coupling coefficient α and of the network dimensions M and N. As was found, a cluster structure, chaotic in time and space, is observed in large lattices in the case of nonzero coupling, similar to that observed in globally coupled networks, e.g., [3].

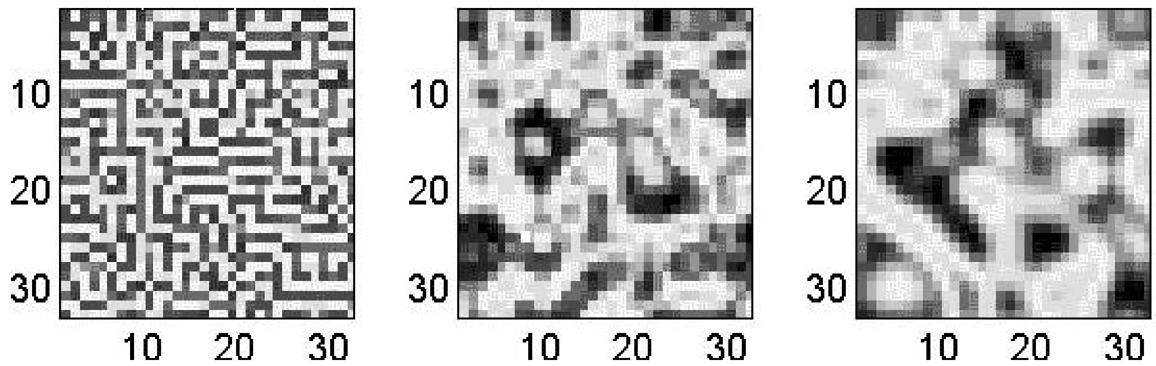


Fig. 1 Cluster structure at the parameter of internal coupling a = 0.2, 0.5, and 0.9. The brightness of a cell corresponds to the value of the corresponding node map.

The clusters are locally synchronous network regions. The nodes of a cluster behave nearly synchronously for some time, though their motion is chaotic in time domain. The cluster boundaries are permanently blurred; some clusters disappear, and others are born. The cluster dimensions depend on the value of the coupling α between the nodes, as it can be seen from Fig. 1, where the snapshots for a 32×32 lattice are presented. As α increases from 0 to 1, average cluster dimensions slightly increase (from one to approximately five), but in large lattices, total spatial synchronization cannot be observed.

However, in small matrices, 5×5 or smaller, total synchronization of the lattice nodes is observed. In order to explain this effect, consider the necessary conditions for the stability of the spatially uniform mode using the theory developed in [4].

3.1. A method for calculation of the stability of synchronous mode [4]

Consider an ensemble of P coupled 1-D maps. Let the ensemble of P equal coupled maps can be described as follows

$$x_i(t+1)=f\left(\alpha_{ii}x_i(t)+\sum_{j\neq i}\alpha_{ij}x_j(t)\right)=f\left(\sum_{j=1}^{P}\alpha_{ij}x_j\right),\ i=1,\ \dots,\ P, \tag{2}$$

where α_{ij} is the coefficient of the effect of ith map on jth one, and f is the mapping function. We call the matrix $\mathbf{A} = \|\alpha_{ij}\|$ the coupling matrix of the ensemble (2).

Synchronization mode of the ensemble (2) is a mode with the trajectories of the maps satisfying the following condition and stable with respect to small perturbations

$$x_i(t) = x_1(t),\ i = 1, \ldots, P. \tag{3}$$

It can be easily seen that if the sums of all the rows of matrix **A** are equal to one, i.e., for each node the sum of the weights of the surrounding nodes in (2) and the node itself is equal to one, then regardless of the mapping function f, the trajectory $x(t)$ of the isolated map is solution (3) for system (2), so that $x_i(t) = x(t)$, $i = 1, \ldots, P$. Thus, a synchronous mode always exists, and in the system phase space it lies along the main diagonal of the P-dimensional hypercube. Now, all we have to do is to find the conditions for its stability.

If we treat relations (2) as a mapping of space $\Re^P$ into itself, then the Jacobian **J** of this mapping in every point of trajectory (3) is equal to

$$\mathbf{J}(x(t)) = f'(x(t))\mathbf{A}. \tag{4}$$

As was proved in [4], the necessary conditions for the stability of the synchronous mode of ensemble (2) can be directly calculated from the set of the eigenvalues of the matrix **A**. Here, we omit many details associated with the fine structure of the phase space near the attractor of the synchronous mode, with riddled basins, etc (those questions are carefully discussed in [4]), and state the following necessary condition for the synchronous mode stability with respect to small perturbations

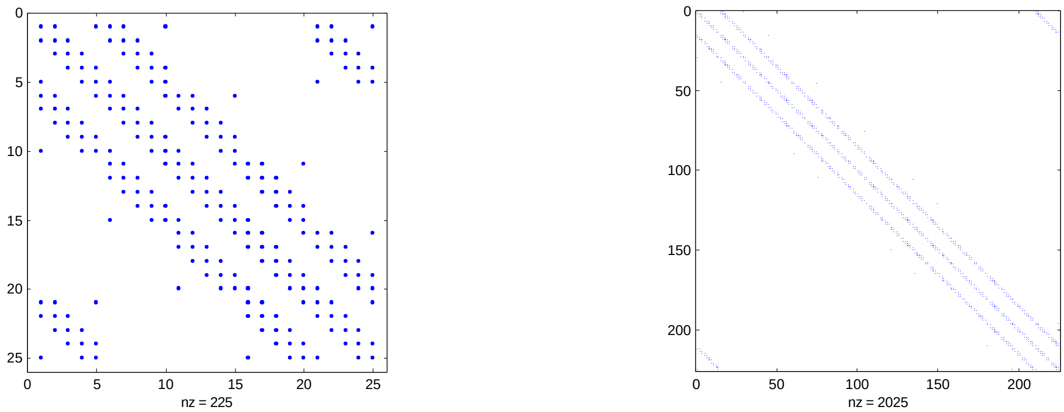
$$\lambda_2 = \lambda + \ln(\rho_{-1}(\mathbf{A})) < 0, \qquad \text{or} \qquad \rho_{-1}(\mathbf{A}) < \exp(-\lambda). \tag{5}$$

Here, λ_2 is the maximum Lyapunov exponent of the motion transversal to the main diagonal, where the system attractor is located. To calculate $\rho_{-1}(\mathbf{A})$, we take the set of eigenvalues of matrix **A** and remove an eigenvalue $\mu = 1$ (which is always present). There may be many eigenvalues equal to 1, but we remove only one of them, and then among the rest, for $\rho_{-1}(\mathbf{A})$ take the magnitude of the largest (by magnitude) eigenvalue.

Relation (5) is universal and does not depend on the concrete type of the dynamic systems constituting the ensemble (2).

3.2 Application of the method to lattices of locally coupled maps

In order to construct the coupling matrix **A** for $M \times N$ lattice of locally coupled maps, we enumerate the lattice nodes from 1 to $P = M \times N$ in a regular way, e.g., row by row from top to bottom. The coupling matrix **A** for P nodes has the size of $P \times P$, and due to only local connections it is sparse for large P, i.e., most of its elements are null. A row of matrix **A** corresponds to a node of the lattice. As follows from the relation for local connections (1), the main diagonal of matrix **A** contains the elements a_{ii} describing the node self-action, i.e., $(1-\alpha)$. Each row i of **A** also contains m coefficients $a_{ij} = \alpha/m$ coupling the corresponding node i to its nearest neighbors. Other elements of the row are null.



(a) *(b)*

Fig. 2 Coupling matrices for (a) 5´5 and (b) 15´15 lattices. Only nonzero elements are shown

In Fig. 2, the coupling matrices for the neighborhood template $\mathbf{T} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and the lattices 5×5 and 15×15 are shown schematically. As can be seen, the coupling matrix becomes more and more sparse with increasing lattice size. Note, that in the case of globally coupled map ensembles the coupling matrix has no zero elements.

The stability of the synchronous mode was calculated according to relation (5). In our case, isolated maps of the lattice are chaotic, so $\lambda > 0$. As follows from (5), synchronization is possible only if $\rho_{-1}(\mathbf{A}) < 1$, hence $\rho(\mathbf{A}) = 1$. This means that the maximum Lyapunov exponent λ_1 of trajectory (3) of system (2) coincides with the maximum Lyapunov exponent λ of trajectory $x(t)$ of the map f.

The numerical research shows that there is no synchronization in the case of weak coupling between the nodes. As the coupling increases, the synchronous mode becomes possible in lattice 3×3 at $\alpha \approx 0.44$, in lattice 4×4 at $\alpha \approx 0.67$, and only at $\alpha \approx 0.96$ in lattice 5×5 (Fig. 3). In bigger lattices global spatial synchronization is impossible at couplings $\alpha \in [0, 1]$. However, these results seem to explain the increase of the typical size of the cluster structure with increasing α, and the average cluster "lifetime" in large lattices.

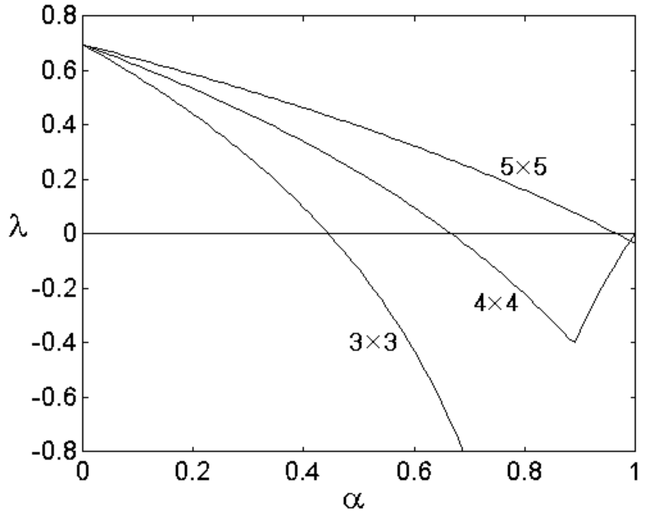


Fig. 3 Second Lyapunov exponent as a function of the local coupling coefficient **a** *for different lattices*

The global synchronous mode can be obtained in large matrices by means of an increase of the size of the neighborhood. Calculations and experiments show that the synchronization becomes possible only when the size of the neighborhood becomes equal to half the size of the lattice, which corrupts the very idea of local connections.

4. Pacemaker-Controlled Model

As we have already found, global synchronization is impossible in large lattices of locally coupled chaotic elements (maps). This result is universal and is valid for any type and dimensionality of the maps. So, in order to provide synchronization, we have to take some special efforts.

Let us add an element to the above lattice, the same chaotic element with the same system parameter, and endow it a special role of pacemaker, i.e., connect this dedicated element to all other lattice elements unidirectionally, with no reverse action. The dynamics of this independent element is described by the mapping equation f, while the behavior of the lattice elements is now described as follows

$$x_{ij}(n+1) = f\left(x_p\beta + (1-\beta)\left[\frac{\alpha}{m}\sum_{s}^{m} x_{is} + x_{ij}(1-\alpha) \right] \right), \quad (6)$$

where x_p is the pacemaker variable. The pacemaker is equally coupled to all lattice nodes, and β is the coupling coefficient. In the case of no action of the pacemaker, i.e., $\beta = 0$, this expression is reduced to the above expression for the lattice of locally coupled chaotic elements (1).

Remember that we look for global synchronous spatial solutions. As follows from the dynamics of coupled discrete-time systems [4], in the case of two same unidirectionally coupled maps, the synchronous mode always exist, which may be proved directly, and the stability condition with respect to small perturbations in linear approximation can be expressed by the inequality

$$(1-\beta) < \exp(-\lambda), \quad (7)$$

where λ is the first, maximum Lyapunov exponent of the first map trajectory. In the case of logistic maps with $\mu = 4$, the Lyapunov exponent $\lambda = \ln 2$, and the condition for the synchronous mode is $\beta \geq 0.5$.

Experiments confirm this theoretical result. We set the initial conditions of the lattice maps to random numbers in the range [0, 1]. The initial condition for the pacemaker was also set random. Then we iterated together the pacemaker $x(t+1) = f(x(t))$ and the lattice equations (6). At $\beta \geq 0.5$ and any coefficient of the internal coupling α, the global spatial synchronous mode was observed.

As follows from the theory [4], if we decrease β to $\beta < 0.5$ in the case of two maps, the synchronous mode is destroyed and can be observed no more. So, we formed a new problem: can the internal local connections preserve the synchronous mode, i.e., what will happen if we decrease the pacemaker effect down to $\beta < 0.5$ in uncoupled (internally) lattice ($\alpha = 0$), and then increase α?

Surprisingly, we found that even in the case of $\beta < 0.5$ even in uncoupled lattice a global synchronous mode is observed. All the lattice maps behave coherently, i.e., an explicitly synchronous mode exists. However, this mode is different from the pacemaker mode. This synchronous mode is also chaotic ($\lambda = 0.52 \div 0.69$) and stable, it attracts the lattice maps from any initial conditions. Local connections do not change this mode, it exists at any α and $\beta \in [0.332..., 0.5]$.

Obviously, the existence of this synchronization could have been explained if we could prove that the oscillation mode in the lattice maps induced by the pacemaker were stable with respect to small perturbations. To investigate the situation, consider the dynamics of the two unidirectionally coupled maps in more details.

4.1 Induced motion in a system of two logistic maps with unidirectional coupling

Consider two unidirectionally coupled logistic maps. Let us vary the coupling coefficient β in the range [0, 0.5] and investigate the stability of the motion induced in the second map with respect to small perturbations in this map (not necessarily with respect to small perturbations in the first map). In order to estimate the stability, we calculate the Lyapunov exponent of the second map trajectory as of an induced motion, i.e., as that of a nonautonomous system

$$\lambda = \ln\left(\lim_{N\to\infty} \left(\prod_{k=1}^{N} \left| \frac{\partial f\left(\beta x_1(k) + (1-\beta) x_2(k)\right)}{\partial x_2} \right|^{1/N} \right) \right) = \ln\left((1-\beta) \lim_{N\to\infty} \prod_{k=1}^{N} \left| \frac{\partial f\left(x_2(k)\right)}{\partial x_2} \right|^{1/N} \right) \qquad (22).$$

Plots of the Lyapunov exponent as a function of the coupling coefficient β are presented in Fig. 4. One can distinguish three main modes of the second map motion related to different ranges of the parameter β. The range ***I*** = [0.5, 1.0] corresponds to the ordinary synchronous mode, and the curve of the Lyapunov exponent behaves as $\lambda^* = \ln(1-\beta) + \lambda$, where $\lambda = \ln 2$ is the Lyapunov exponent of the isolated map. The Lyapunov exponent is negative here, so this synchronous mode is stable.

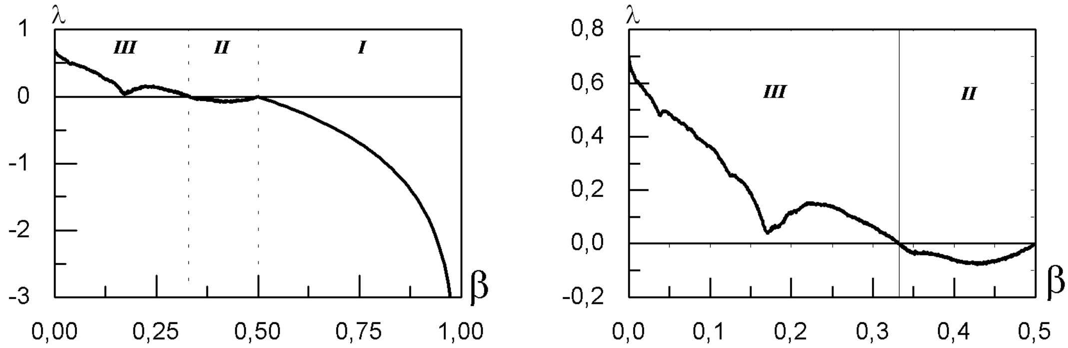
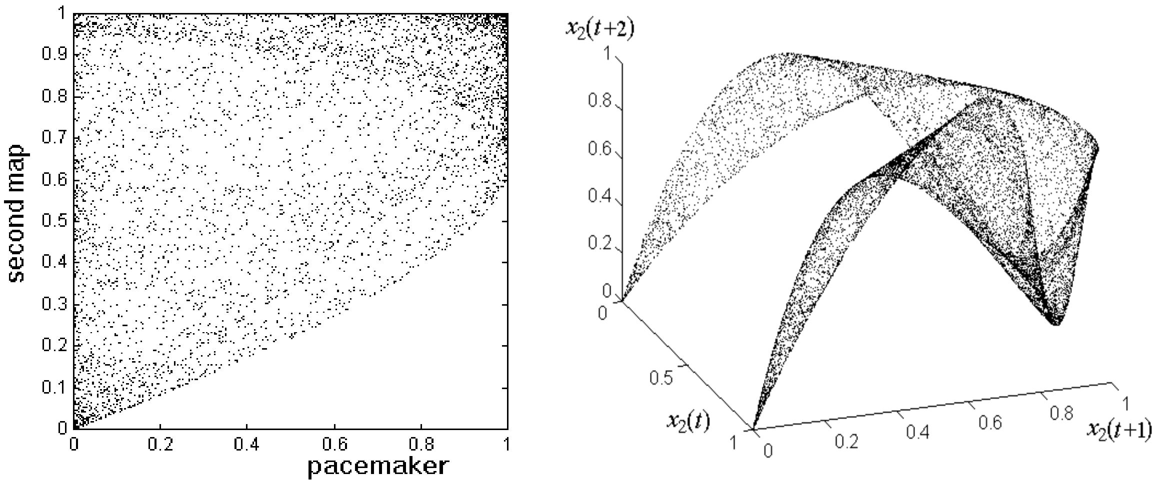


Fig. 4. Lyapunov coefficient of the trajectory induced in the second map.

As β decreases and enters the second range ***II*** = [0.332..., 0.5], the synchronous oscillations are destroyed, however the motion observed now in the second map does not become independent. The induced chaotic oscillations in the second map are stable, which follows from the negative Lyapunov exponent. Thus, a new attractor appears in the phase space of the dynamic system, that attracts trajectories from almost all initial conditions. This attractor does not lie along the main diagonal of the phase space (Fig. 5). We will call this mode "hidden" synchronization, because it is responsible for the synchronous motion of the CCNN in the case when it is not evident. Within all the range ***II*** the form of the attractor remains the same.

Finally, the third type of the motion takes place at $\beta \in$ ***III*** = [0, 0.332...]. The Lyapunov exponent of the induced trajectory becomes positive, an as $\beta \to 0$, the system motion rapidly becomes independent.

The obtained results help us to explain the phenomena in chaotic CNN composed of 1-D maps.



(a) *(b)*

*Fig. 5 Nondiagonal attractor of zone **II**. (a) the phase portrait, (b) the attractor reconstructed in a time-delay space*

4.2 Coupling of chaotic elements and spatial synchronization

In the motion of the lattice of logistic maps, driven by the pacemaker, we can distinguish three different modes: the mode of total synchronization at $\beta \in [0.5, 1]$, the mode of spatially synchronous lattice observed at $\beta \in [0.332..., 0.5]$ and any couplings of the lattice elements ("hidden" synchronization), and desynchronization mode at low β.

Now we form a new problem: can local connections of the lattice nodes preserve the spatial synchronous lattice mode on the edge of ***II*** and ***III*** zones, i.e., at $\beta = 0.332...$?

Unfortunately, we cannot use here the approach described in Section 3.1, because the main assumption (4) is invalid here due to different modes of the pacemaker and the lattice. So, we tried to get the answer to the problem experimentally. We set the initial conditions of the lattice nodes and the pacemaker to random numbers and iterated the system equations for different α and β.

We have found experimentally that the spatial synchronous mode survives in a narrow ribbon of $\beta \in [0.31\dots, 0.332...]$ and nonzero α. The closer β to the boundary of ***II*** and ***III*** zones, the faster is the convergence to the global synchronous mode. Typical values of the convergence time are about 100 steps for the value of local connections α close to 1. As α is decreased and β lowered toward $\beta = 0.31\dots$, the convergence time increases to tens of thousand steps. Below $\beta = 0.31\dots$, the system becomes desynchronized, though near this value and at very weak connections local connections, intermittency between the spatial chaos and spatial synchronous mode is explicitly observed.

Conclusions

We proposed a numerical description of the dynamics of spatially synchronous mode of chaotic CNNs (cellular nonlinear networks), and found the range of its stability as a function of the network size and the strength of local connections.

In a model of pacemaker-controlled CNN, we have found a spatial synchronous mode induced by the external pacemaker, though this synchronous mode is different from the pacemaker's mode.

It is shown that on the edge of spatially synchronous and desynchronized mode regions, the presence of local connections preserves the stability of the spatially homogeneous chaotic mode.

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