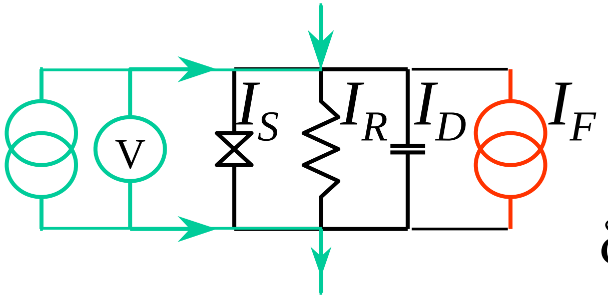
Macroscopic Quantum Effects in Josephson junctions

Edward Goldobin

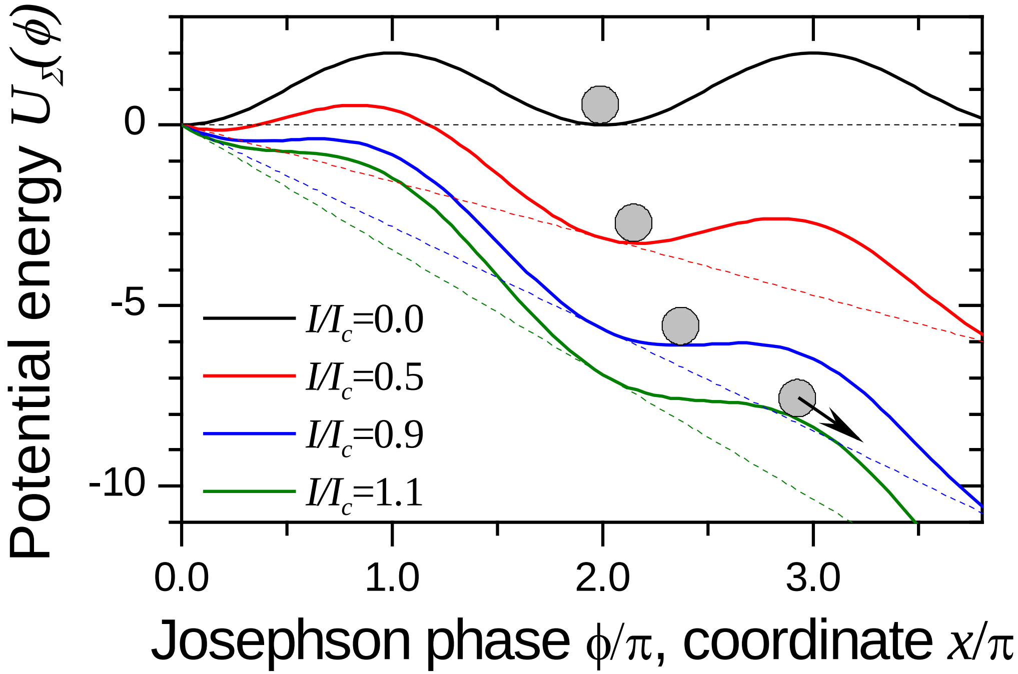
University of Tübingen, Germany

Thermal fluctuations



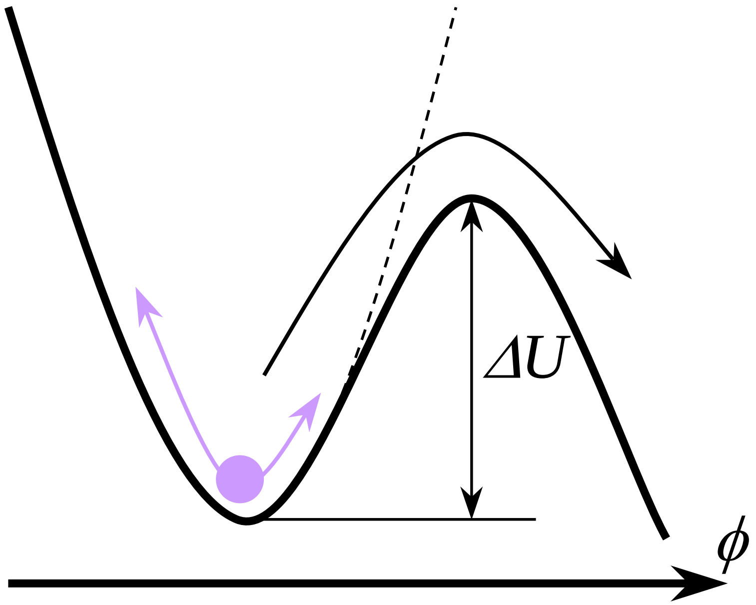
$$I + I_F(t) = I_c \sin(\phi) + \frac{\Phi_0}{2\pi R}\dot{\phi} + \frac{\Phi_0 C}{2\pi}\ddot{\phi}$$

δ-correlated (white) noise: $\langle I_F(0) I_F(t)\rangle = \frac{4k_B T}{R}\delta(t)$



Particle may escape at $I<I_c$ ($\gamma<1$) due to thermal fluctuations!

Thermal escape of the phase



$$\omega_p(\gamma) = \omega_p(0)\sqrt[4]{1-\gamma^2}$$

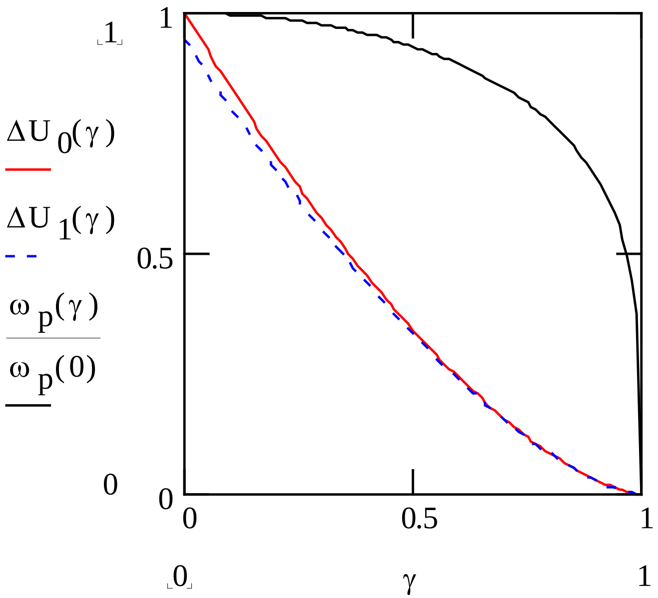
$$\Delta U(\gamma) = E_J 2\left[\sqrt{1-\gamma^2} - \gamma\arccos(\gamma)\right]$$

$$\Delta U(\gamma) \approx E_J \frac{4\sqrt{2}}{3}(1-\gamma)^{\frac{3}{2}}, \quad \gamma \to 1$$

$$P(E) = \exp(-E/k_B T)$$

Escape rate: $\Gamma_K = \frac{\omega_p}{2\pi}\exp\left(-\frac{\Delta U}{k_B T}\right)$

Estimations:



$I_c \equiv 100\cdot\mu A \qquad C \equiv 4\cdot\mu F\cdot cm^{-2}\cdot 10\cdot\mu m\cdot 10\cdot\mu m$

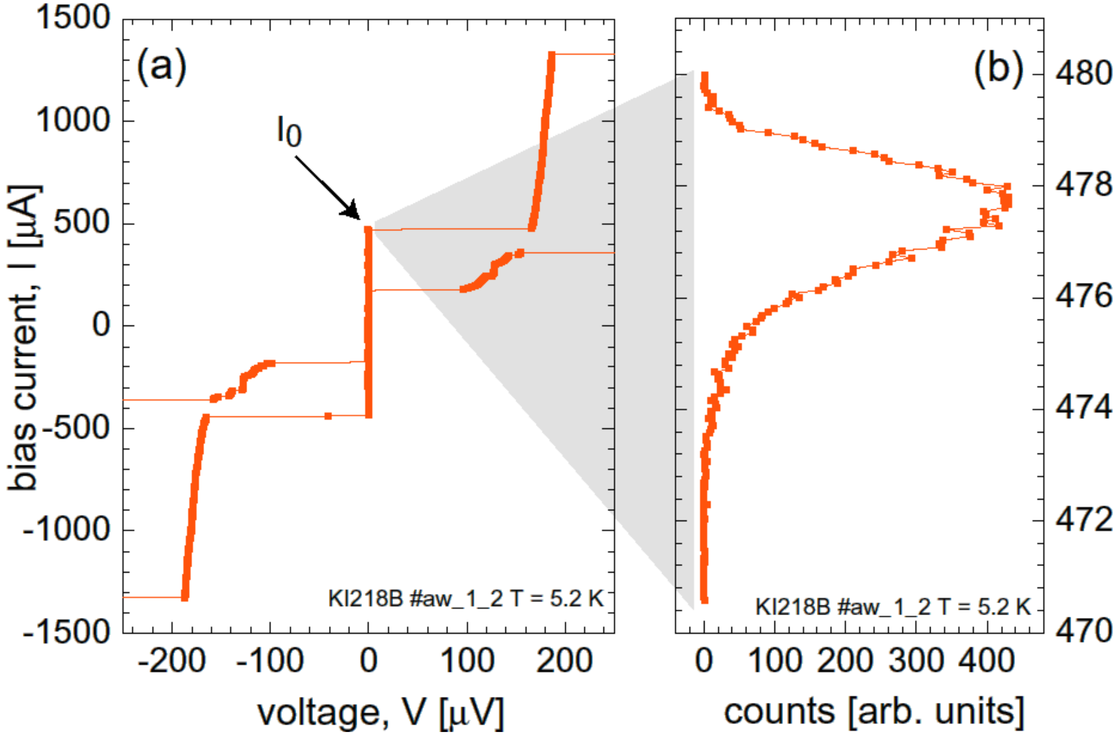
$\frac{E_J}{k_B} = 2.384\cdot 10^3\cdot K \qquad \frac{\omega_p(0)}{2\cdot\pi} = 43.865\cdot GHz$

$\gamma := 0.99 \quad \Delta U_1(\gamma)\cdot\frac{E_J}{k_B} = 2.247\cdot K \qquad \frac{\omega_p(\gamma)}{2\cdot\pi} = 16.475\cdot GHz$

$\Gamma(\gamma, 1\cdot K) = 1.741\cdot GHz \qquad \Gamma(0.95, 1\cdot K) = 0.3\cdot Hz$

$\Gamma(\gamma, 0.3\cdot K) = 9.193\cdot MHz \quad \Gamma(\gamma, 0.1\cdot K) = 2.862\cdot Hz$

Measurement of escape rate



$N \sim 10^4$-10^6 samples: $I_1..I_N$ divided in bins ΔI.

$$dN(t) = -\Gamma(t)N(t)dt$$

$N(t)$ particles survived up to time t.

$$\Gamma(t_i) = -\frac{1}{\Delta t}\ln\left(\frac{N(t_{i+1})}{N(t_i)}\right)$$

flight time vs. current

$$\Gamma(I) = \frac{1}{\Delta I}\frac{dI}{dt}\ln\left(\frac{N(I_{i+1})}{N(I_i)}\right)$$

$$N(I_i) = \sum_{j=1}^{i} n_j$$

Note:

- I_c i.e. ω_p are not known exactly.
-

Thermal escape rate

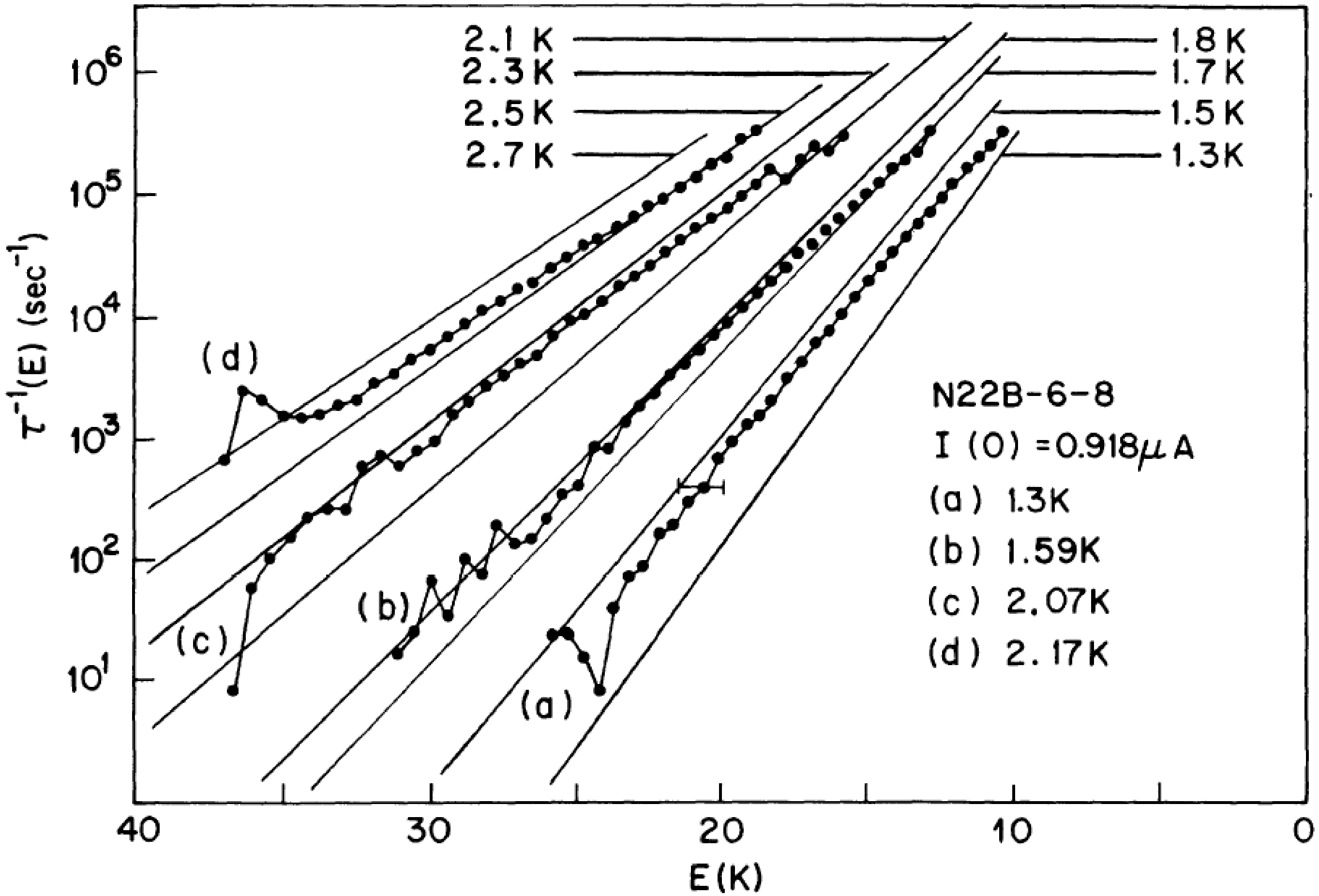
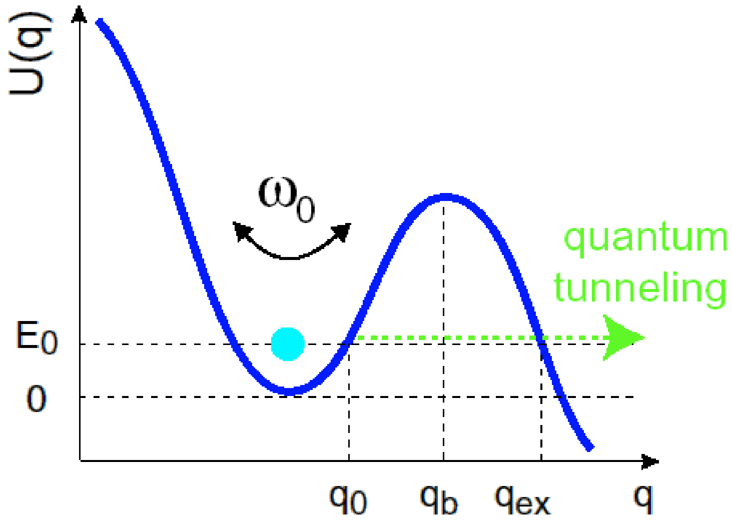


FIG. 6. Data points: $\log_{10}\tau^{-1}(E)$ vs E for junction of Fig. 4. Solid lines: theoretical dependence of $\log_{10}\tau^{-1}(E)$ on E for $\bar{\omega}_J/2\pi = 6.79\times10^8\ [I_c(1-x^2)^{1/2}]^{1/2}$ with I_c measured in μA. Note that generally the temperatures chosen for the theoretical plots exceed the bath temperature by ~0.1–0.2 K.

Fulton et al., PRB **9**, 4760 (1974)

Quantum escape



Does our particle (Jopsephson phase) can be treated as a quantum particle?
If yes,

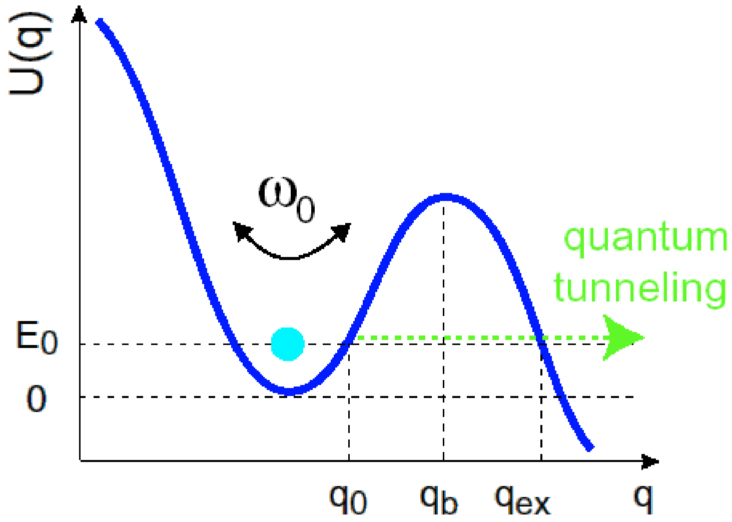
- What is its mass?
- How large is our Plank constant?

$$\Gamma_{qu} = \mathrm{const.}\,\omega_0 \left(\frac{U_0}{\hbar\omega_0}\right)^{1/2} \exp\left(-2\int_{q_0}^{q_{ex}} \frac{1}{\hbar}\sqrt{2m(U(q)-E_0)}\,dq\right)$$

$$\Gamma_{qu} = \omega_0 \left(\frac{60U_0}{\hbar\omega_0}\right)^{1/2} \left(\frac{18}{5\pi}\right)^{1/2} \exp\left(-\frac{36}{5}\frac{U_0}{\hbar\omega_0}\right)$$

Quantum escape rate does not depend on temperature!

Dissipative quantum tunneling



Damping *a*:

$$\Gamma_{qu} = A\exp(-B)$$

$$A = \sqrt{60}\omega_0 \left(\frac{B}{2\pi}\right)^{1/2} (1 + \mathcal{O}(a)),$$

$$B = \frac{36U_0}{5\hbar\omega_0}(1 + 1.74a + \mathcal{O}(a^2)),$$

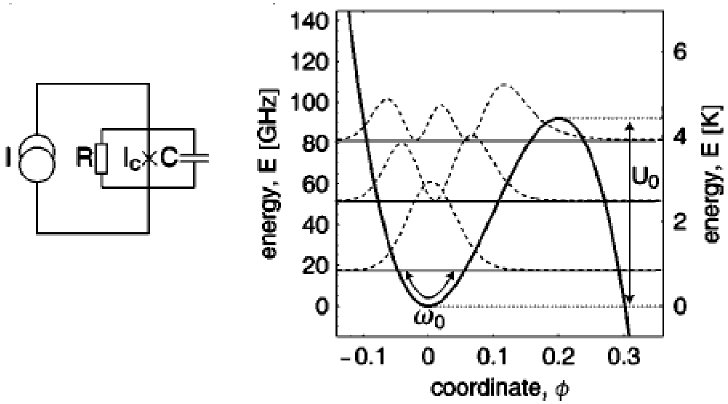
Tunnelling rate reduces!

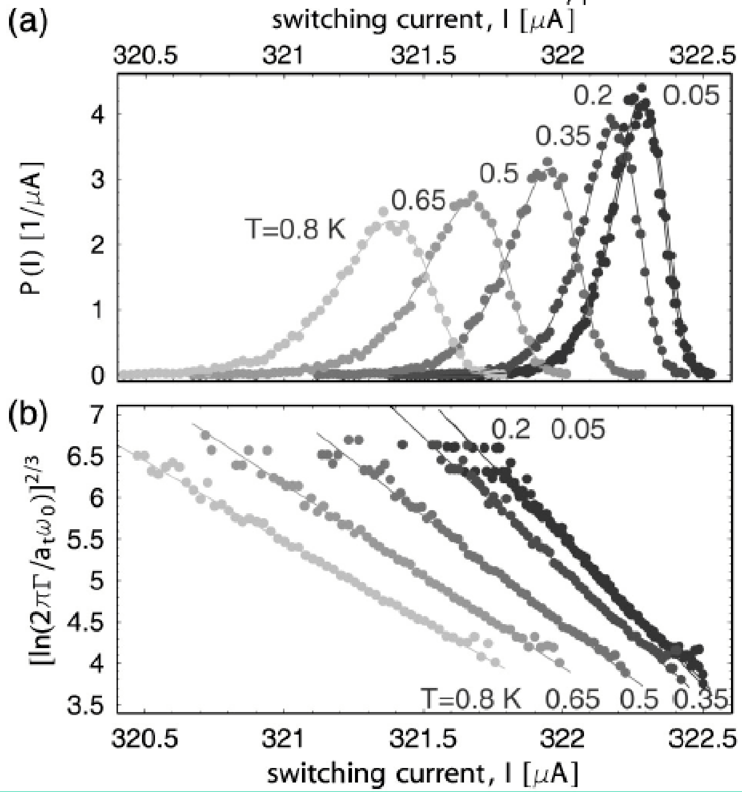
Caldeira, Legget, PRL **46**, 211 (1981)

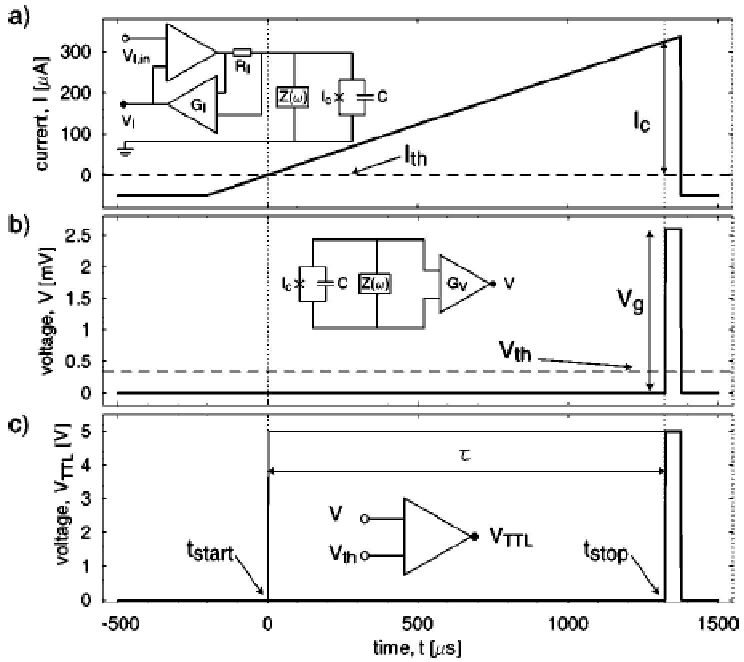
Caldeira, Ann. of Phys. **149**, 374 (1983)

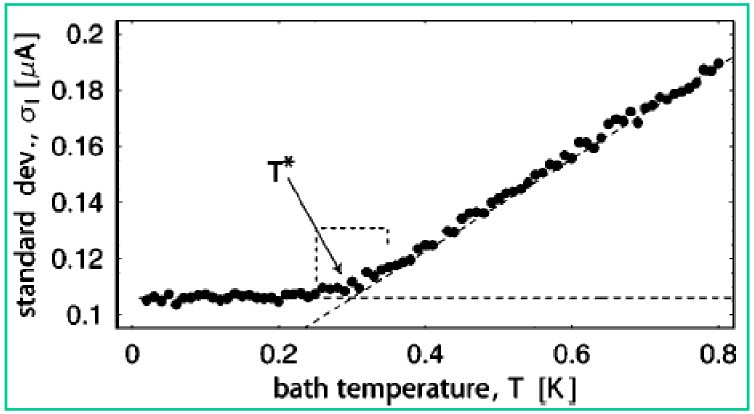
Legget, NATO ASI, Plenum Press, New York, (1984)

Experiments





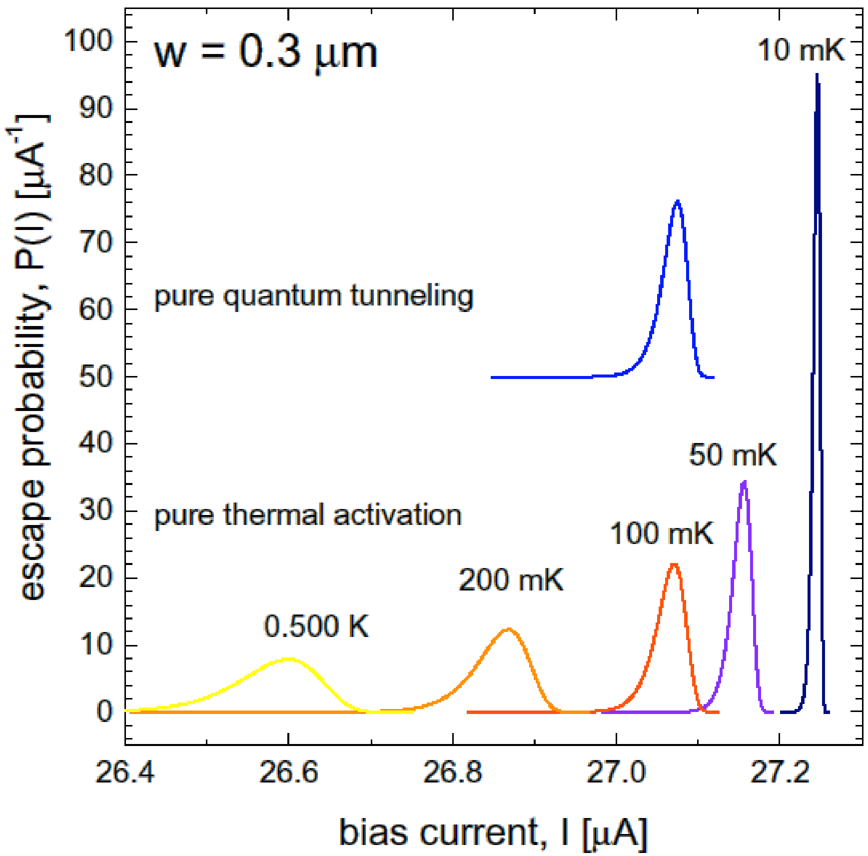


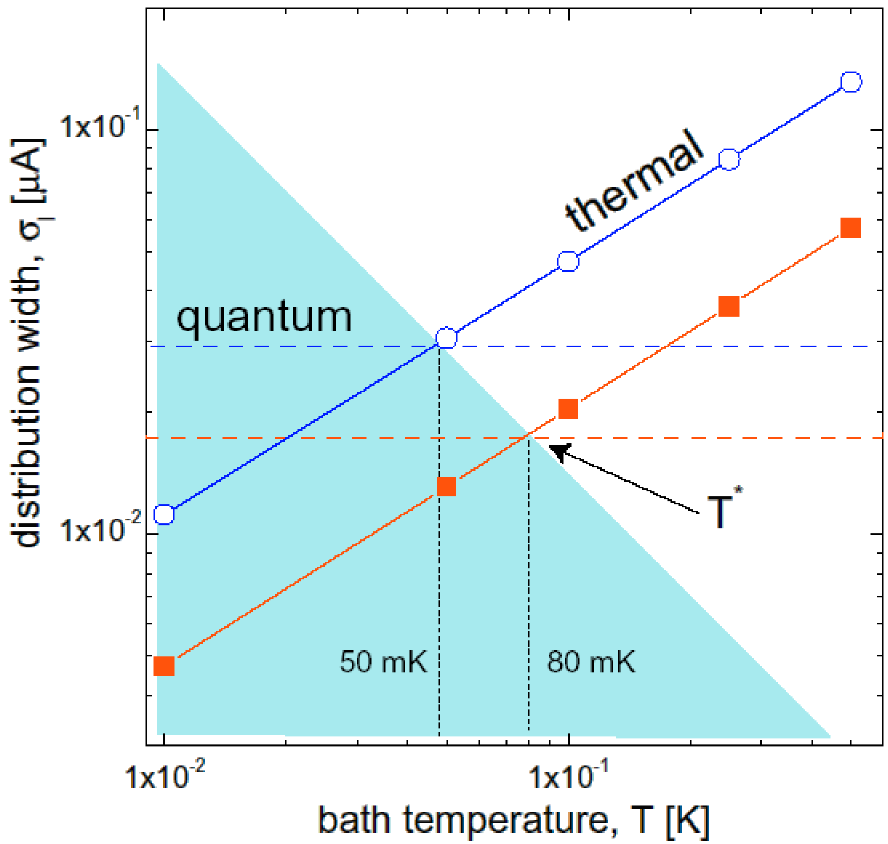


A. Wallraff, Rev. Sci. Instr. **74**, 3740 (2003)

Thermal+Quantum escape

Thermal-Quantum Crossover Temperature *T**





w=3μm   w=0.3μm

A. Wallraff, Ph.D. Thesis (2001)

QM formulation: phase & charge

1.1 QM description

$$L = C\left(\frac{\Phi_0}{2\pi}\right)^2 \frac{\dot{\phi}^2}{2} - E_J(1 - \cos\phi)$$

Canonical variables q and p:

$$\begin{aligned} q &= \phi; \\ p_\phi &= \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{\phi}} = C\left(\frac{\Phi_0}{2\pi}\right)^2 \dot{\phi} = \frac{\Phi_0}{2\pi} Q, \end{aligned}$$

where Q is the charge at the capacitor of JJ.

Quantum mechanics. $\hat{p}_\phi$ in ϕ-representation

$$\hat{p}_\phi = -i\hbar \frac{\partial}{\partial q} = -i\hbar \frac{\partial}{\partial \phi}$$

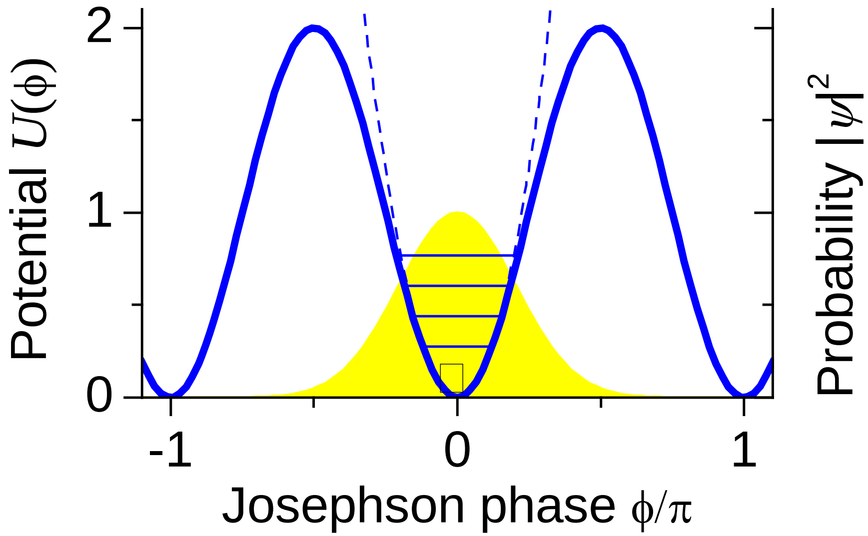
Commutation relations:

$$-i\hbar = [\hat{p}_\phi, \hat{q}] = \frac{\Phi_0}{2\pi}[\hat{Q}, \hat{\phi}]$$

$$[\hat{Q}, \hat{\phi}] = -2ie$$

Uncertainty relation!

Phase in a harmonic potential



$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{m\omega_0^2}{2}x^2\psi=E\psi$$

$$-\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial\phi^2}+\frac{M\omega_0^2}{2}\phi^2\psi=E\psi$$

From classic *H*, the "mass" *M* is:

$$M:M=\left(\frac{\Phi_0}{2\pi}\right)^2 C\,\mathrm{kg}\times\mathrm{m}^2$$

$$\langle\phi^2\rangle=\frac{\hbar}{M\omega_0}=\left(\frac{2\pi}{\Phi_0}\right)^{\frac{3}{2}}\frac{\hbar}{\sqrt{I_cC}}=2\sqrt{2}\sqrt{\frac{E_C}{E_J}}$$

$$E_C\equiv\frac{e^2}{2C},\quad E_J=\frac{\Phi_0 I_c}{2\pi}$$

$$\langle\phi^2\rangle=\left(\frac{2\pi}{\Phi_0}\right)^{\frac{3}{2}}\frac{\hbar}{\sqrt{j_cC'}}\frac{1}{A}$$

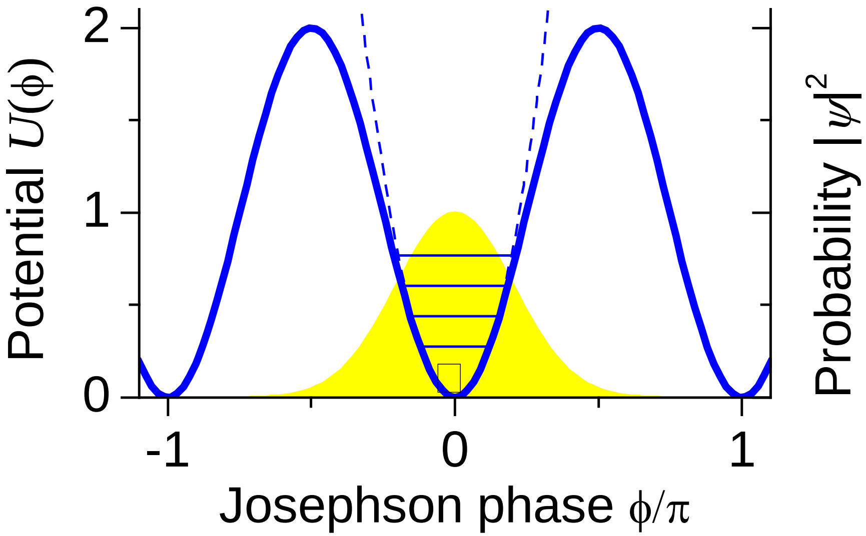
Small junctions!

Example: $A=0.5\times0.5\,\mu\mathrm{m}^2,\ j_c=100\,\mathrm{A/cm}^2,\ C'=4\,\mu\mathrm{F/cm}^2$
gives $I_c=0.25\,\mu\mathrm{A}$

$$\langle\phi^2\rangle_{\gamma=0}\sim0.35$$

$$\langle\phi^2\rangle_{\gamma=0.99}\sim0.94$$

Phase in cos-potential



Phase in cos-potential

$$-\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial\phi^2} + \frac{M\omega_0^2}{2}(1-\cos\phi)\psi = E\psi$$

Mathieu Equation:

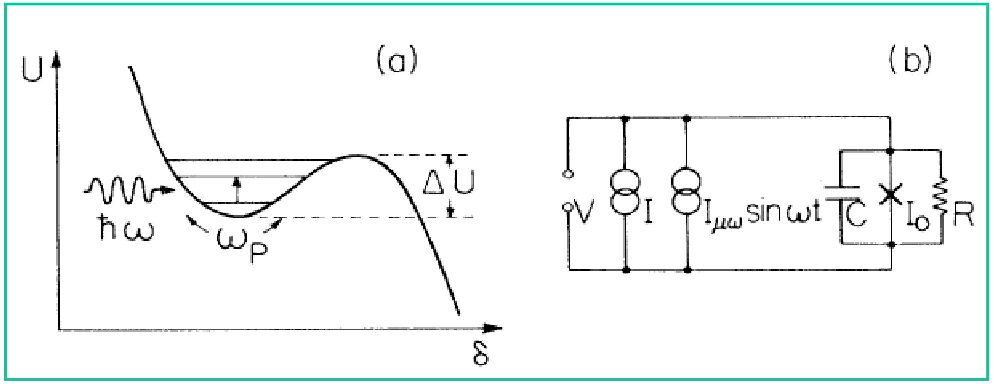
$$\frac{d^2\psi}{dx^2} + [a - 2q\cos(2x)]\psi = 0$$

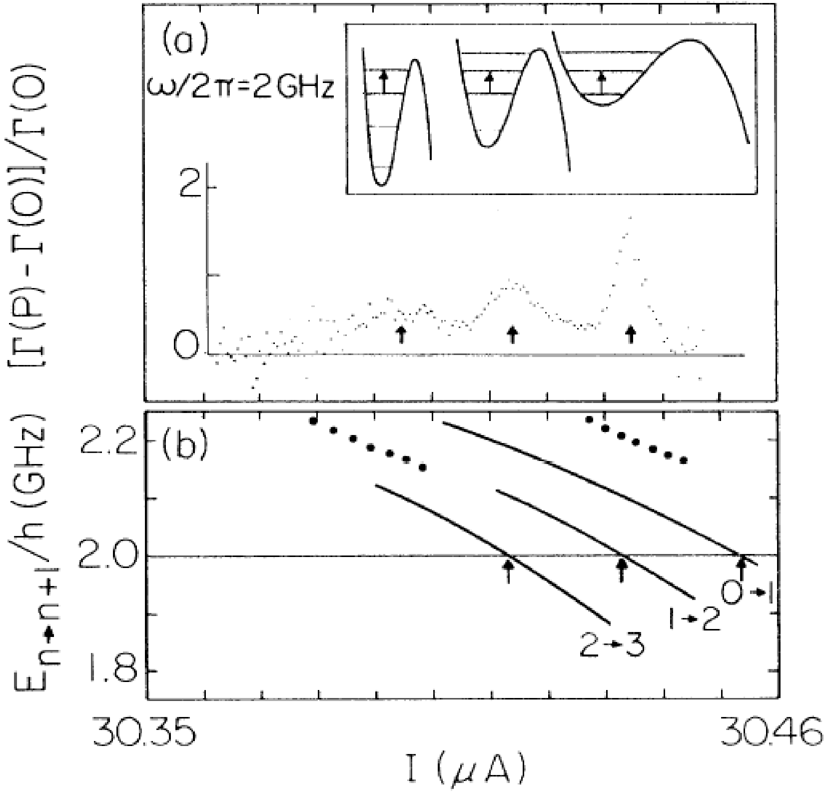
$$\hbar\omega_{01} \neq \hbar\omega_{12} \neq \hbar\omega_{23}\ldots$$

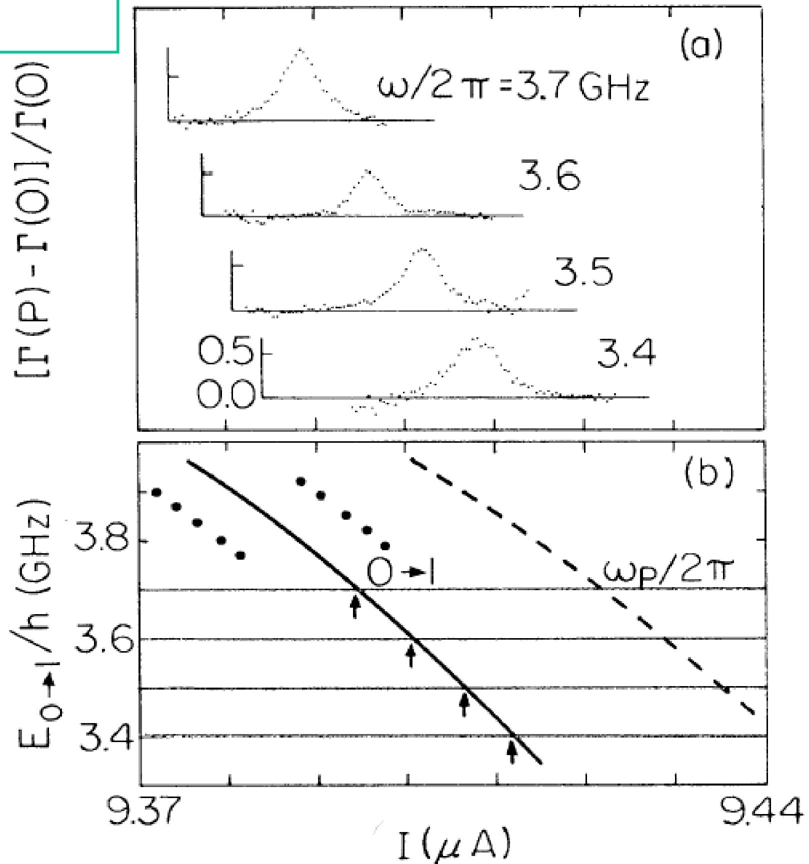
Not equidistant level spacing!
(important for qubits, see below)

Baker et al., The Pendulum, ch.8, Quantum Pendulum, OUP 2005.

Energy Levels Quantization



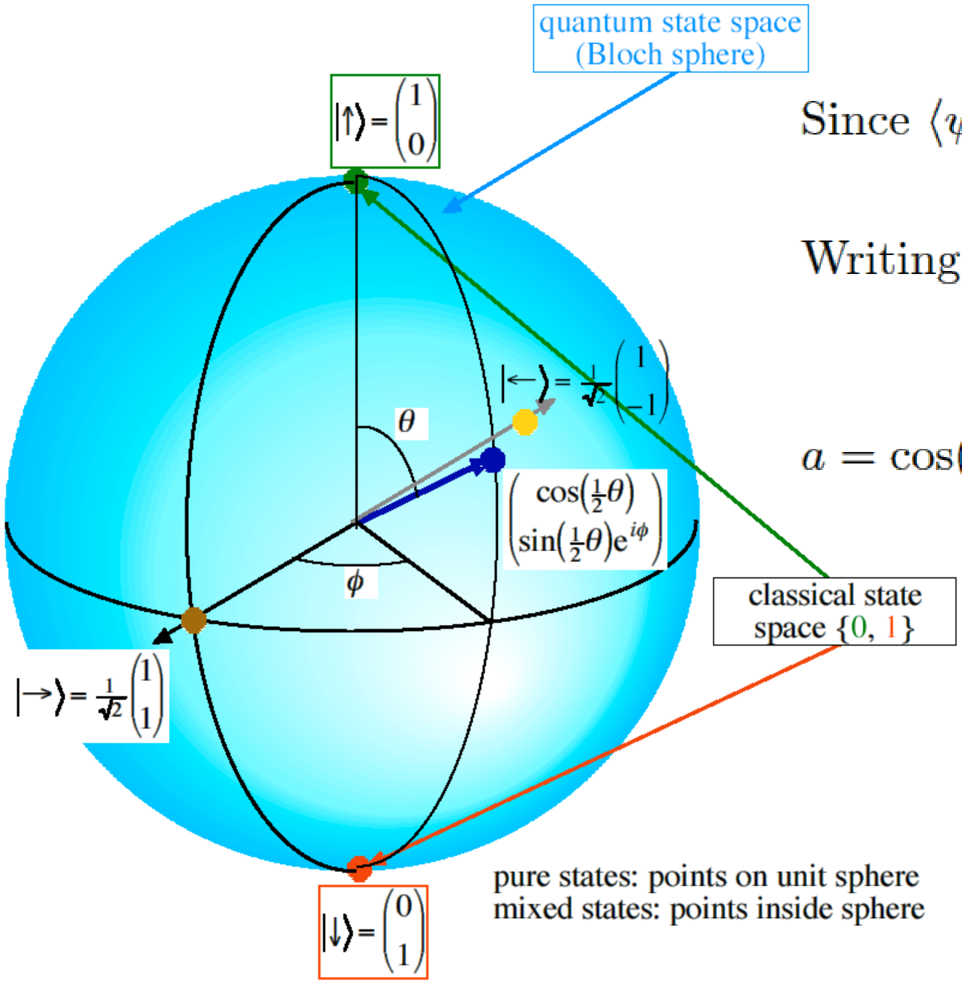




Martinis et al. PRL **55**, 1543 (1985).

Bloch sphere 1

• geometric representation of qubit (2-level system) states as a pont on the surface of a unit sphere
• qubit manipulations can be neatly described



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Since $\langle\psi|\psi\rangle = 1$

$$|\alpha|^2 + |\beta|^2 = 1$$

Writing $\alpha = ae^{i\varphi_\alpha}$ and $\beta = be^{i\varphi_\beta}$, we get

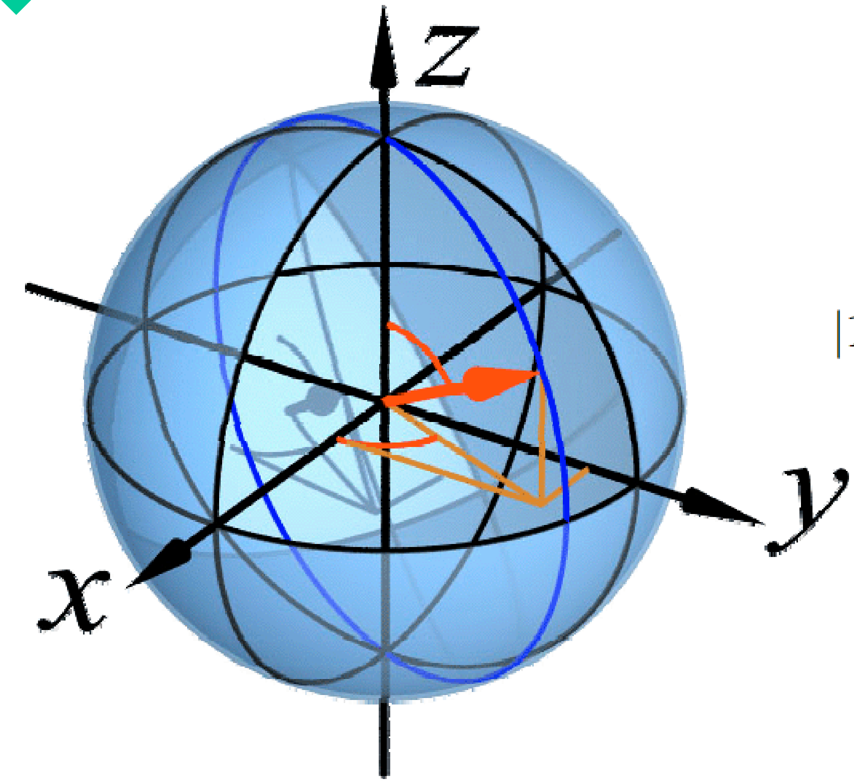
$$|\psi\rangle = e^{i\varphi_\alpha}\left[a|0\rangle + be^{i\varphi}|1\rangle\right]$$

$a = \cos(\theta/2)$, $b = \sin(\theta/2)$

$$\begin{aligned}\alpha &= \cos(\theta/2);\\ \beta &= e^{i\varphi}\sin(\theta/2),\end{aligned}$$

$$|\psi\rangle \equiv \begin{pmatrix}\cos\varphi\sin\theta\\ \sin\varphi\sin\theta\\ \cos\theta\end{pmatrix}$$

Bloch Sphere 2: Examples



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\begin{aligned} \alpha &= \cos(\theta/2); \\ \beta &= e^{i\varphi}\sin(\theta/2), \end{aligned}$$

$$|\psi\rangle \equiv \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}$$

$$|0\rangle \equiv |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|1\rangle \equiv |\downarrow\rangle = \begin{pmatrix} 0 \\ e^{i\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \hat{\sigma}_z|1\rangle = -1|1\rangle$$

$$\frac{1}{\sqrt{2}}|0\rangle + e^{i\varphi}\frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

Orthogonality.

$$\begin{aligned} R_x(\vartheta) &= e^{i\vartheta\sigma_x/2} = \begin{bmatrix} \cos(\vartheta/2) & -i\sin(\vartheta/2) \\ -i\sin(\vartheta/2) & \cos(\vartheta/2) \end{bmatrix}; \\ R_y(\vartheta) &= e^{i\vartheta\sigma_y/2} = \begin{bmatrix} \cos(\vartheta/2) & -\sin(\vartheta/2) \\ +\sin(\vartheta/2) & \cos(\vartheta/2) \end{bmatrix}; \\ R_z(\vartheta) &= e^{i\vartheta\sigma_z/2} = \begin{bmatrix} e^{-i\vartheta/2} & 0 \\ 0 & e^{+i\vartheta/2} \end{bmatrix}, \end{aligned}$$

Rabi oscillations 1

Apply transverse *circulary polarized magnetic field*, i.e.

$$B_x = h\cos(\omega t), \quad B_y = -h\sin(\omega t).$$

The Hamiltonian of the system is

$$\hat{H} = -g\hbar\mathbf{B}\hat{\mathbf{S}} = -g\hbar\left(B_x\hat{S}_x + B_y\hat{S}_y + B_z\hat{S}_z\right)$$

$$\hat{H} = -\hbar\omega_0\hat{S}_z - \frac{g\hbar}{2}\left(B_+\hat{S}_- + B_-\hat{S}_+\right),$$

where ω_0 is eigen precession frequency,

$$B_\pm = B_x \pm iB_y = he^{\mp\omega t}$$

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y = |0\rangle\langle 1|$$

$$\hat{S}_- = \hat{S}_x - i\hat{S}_y = |1\rangle\langle 0|$$

Inserting this into $\hat{H}$, we get

$$\hat{H} = -\frac{\hbar}{2}\left\{\omega_0(|0\rangle\langle 0| - |1\rangle\langle 1|) + \Omega\left(e^{i\omega t}|0\rangle\langle 1| + e^{-i\omega t}|1\rangle\langle 0|\right)\right\},$$

where $\Omega = gh$ is *the Rabi frequency*. It describes the transitions between the states $|0\rangle$ and $|1\rangle$ inder the action of resonant field.

Rabi, Phys. Rev. **51**, 652 (1937).

Rabi oscillations 2

Inserting this into $\hat{H}$, we get

$$\hat{H} = -\frac{\hbar}{2}\left\{\omega_0(|0\rangle\langle 0| - |1\rangle\langle 1|) + \Omega\left(e^{i\omega t}|0\rangle\langle 1| + e^{-i\omega t}|1\rangle\langle 0|\right)\right\},$$

where $\Omega = gh$ is *the Rabi frequency*. It describes the transitions between the states $|0\rangle$ and $|1\rangle$ inder the action of resonant field.

If we substitute $\Psi = \alpha(t)|0\rangle + \beta(t)|1\rangle$ into the Schrdinger equation

$$i\hbar\dot{\Psi} = \hat{H}\Psi$$

using resonant condition $\omega = \omega_B = gB$, after some algebra, we will find

$$\alpha(t) = \alpha(0)\cos\left(\frac{\Omega t}{2}\right) + i\beta(0)\sin\left(\frac{\Omega t}{2}\right); \quad (1)$$

$$\alpha(t) = i\alpha(0)\sin\left(\frac{\Omega t}{2}\right) + \beta(0)\cos\left(\frac{\Omega t}{2}\right), \quad (2)$$

Rabi, Phys. Rev. **51**, 652 (1937).

Rabi oscillations 3

Example 1 (π-pulse, $t_1 = \pi/\Omega$):

$$\begin{pmatrix} \alpha(0) \\ \beta(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha(t_1) \\ \beta(t_1) \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

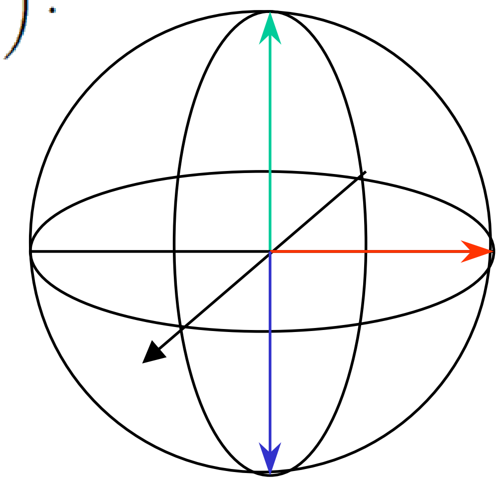
Example 2 (another π-pulse, $t_1 = \pi/\Omega$):

$$\begin{pmatrix} \alpha(0) \\ \beta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha(t_1) \\ \beta(t_1) \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix}.$$

Example 3 ($\pi/2$-pulse, $t_1 = \pi/2\Omega$):

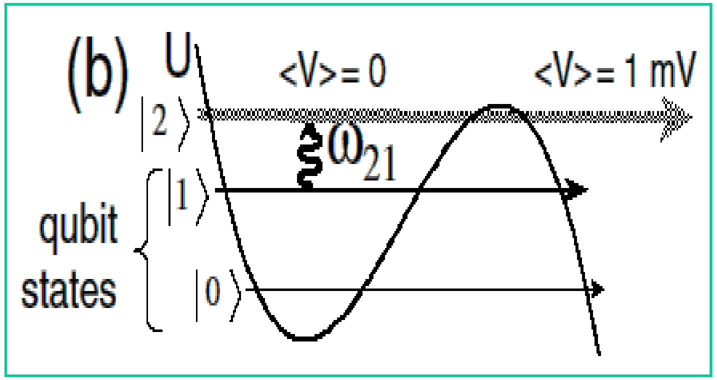
$$\begin{pmatrix} \alpha(0) \\ \beta(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha(t_1) \\ \beta(t_1) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) \\ i\sin\left(\frac{\pi}{4}\right) \end{pmatrix}.$$

$$|\alpha(t_1)|^2 = \frac{1}{2}, \quad |\beta(t_1)|^2 = \frac{1}{2}.$$



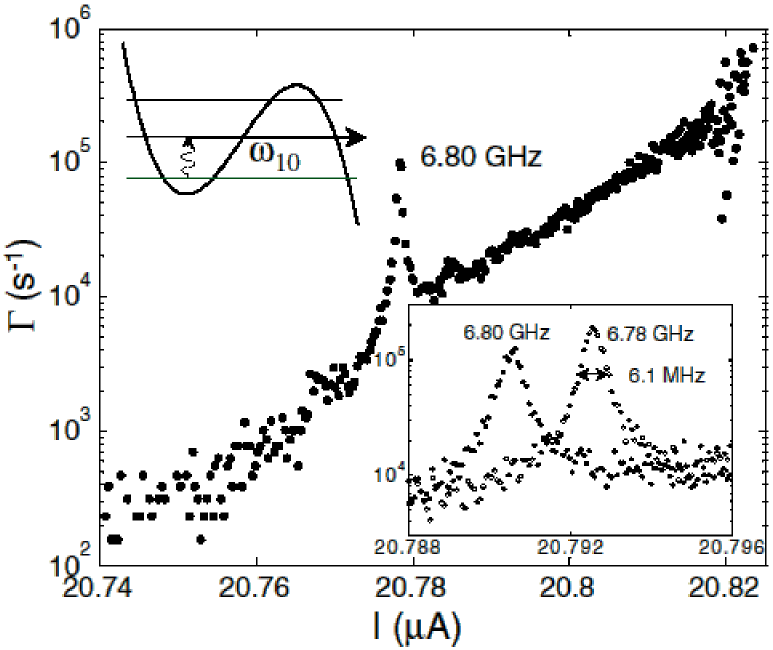
Rabi, Phys. Rev. **51**, 652 (1937).

Superposition of |0⟩ and |1⟩



$$I(t) = I_{dc} + \delta I_{dc}(t) + I_{\mu wc}(t)\cos\omega_{10}t + I_{\mu ws}(t)\sin\omega_{10}t.$$

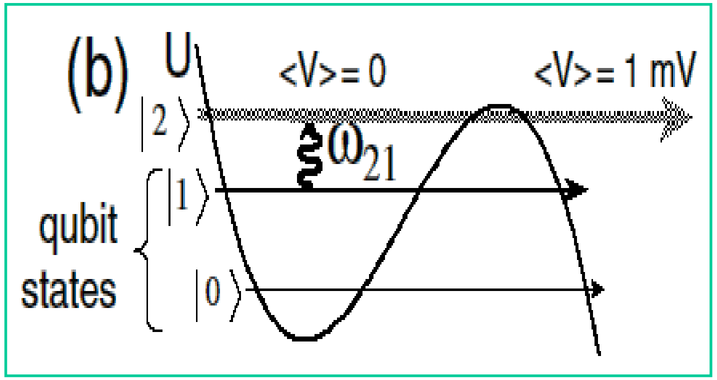
$$H = \hat{\sigma}_x I_{\mu wc}(t)\sqrt{\hbar/2\omega_{10}C}/2 + \hat{\sigma}_y I_{\mu ws}(t)\sqrt{\hbar/2\omega_{10}C}/2 + \hat{\sigma}_z \delta I_{dc}(t)(\partial E_{10}/\partial I_{dc})/2,$$



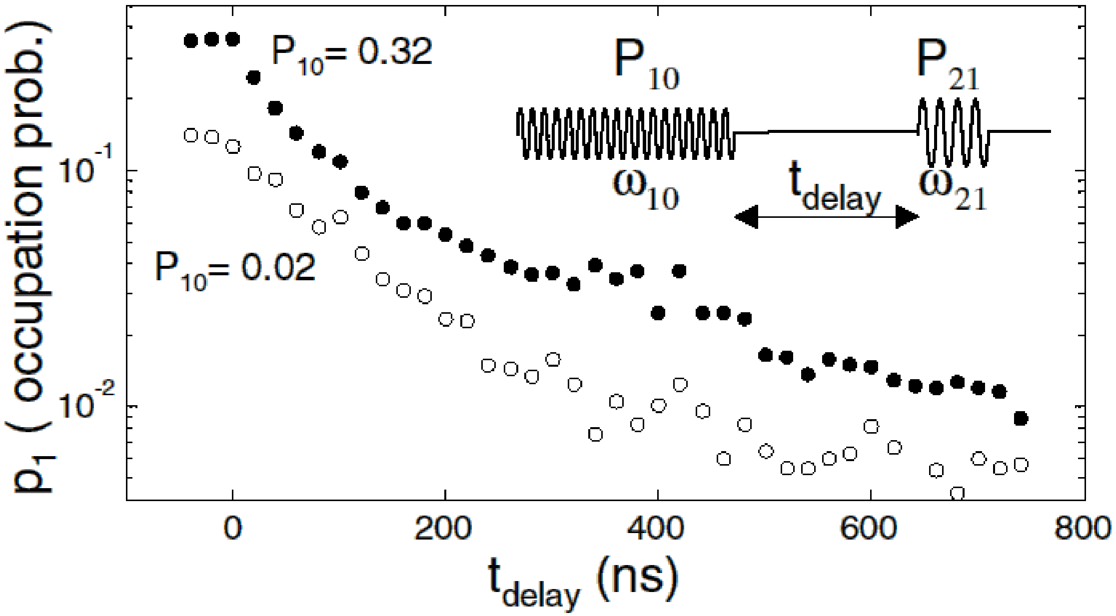
$\Delta\omega_{10}/\omega_{10}$ --> Q~1000

Martinis et al. PRL **89**, 117901 (2002).

Superposition of |0⟩ and |1⟩



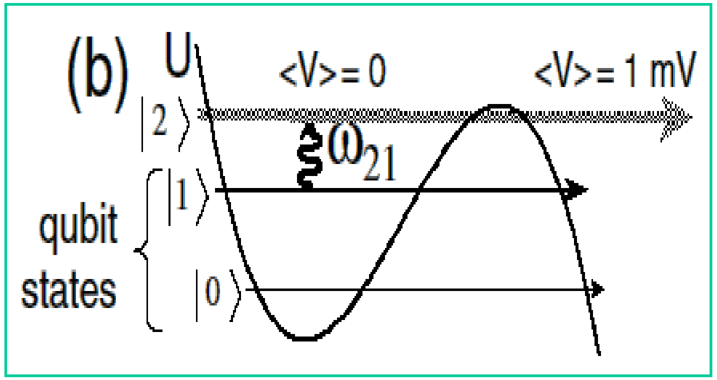
Relaxation rate (20ns, then 300 ns):



$$I(t) = I_{\rm dc} + \delta I_{\rm dc}(t) + I_{\mu \rm wc}(t) \cos\omega_{10} t + I_{\mu \rm ws}(t) \sin\omega_{10} t.$$

$$H = \hat{\sigma}_x I_{\mu \rm wc}(t)\sqrt{\hbar/2\omega_{10}C}/2 + \hat{\sigma}_y I_{\mu \rm ws}(t)\sqrt{\hbar/2\omega_{10}C}/2 + \hat{\sigma}_z \delta I_{\rm dc}(t)(\partial E_{10}/\partial I_{\rm dc})/2,$$

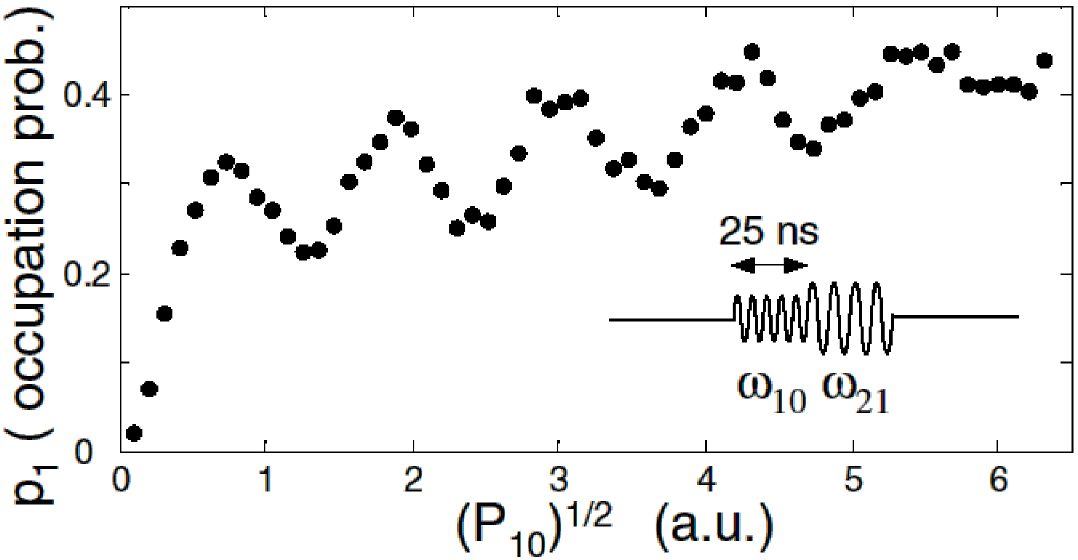
Rabi oscillations in phase qubit



Rabi oscillations:

- prepare coherent mixture by short pulse
- RabiPhase ~ $P_{01} * T_{01}$.
- apply measurement pulse
- Since coherence is short we measure $p(P_{01})$

$$H = \hat{\sigma}_x I_{\mu\mathrm{wc}}(t)\sqrt{\hbar/2\omega_{10}C}/2 + \hat{\sigma}_y I_{\mu\mathrm{ws}}(t)\sqrt{\hbar/2\omega_{10}C}/2 + \hat{\sigma}_z \delta I_{\mathrm{dc}}(t)(\partial E_{10}/\partial I_{\mathrm{dc}})/2,$$



Martinis et al. PRL **89**, 117901 (2002).